CSE 312: Foundations of Computing II

Section 10: Final Review Solutions

0. Sorted Hands

A bridge deck consists of 52 cards divided into 4 suits of 13 ranks each. A bridge hand consists of 13 cards from a bridge deck. Suppose that the bridge cards are well shuffled and dealt. What is the probability that your bridge hand is already sorted when you pick it up, given that you have been dealt at least two cards in each of the 4 suits? By "sorted" I mean that the cards of any one suit are adjacent to each other, and the cards of each suit are sorted by rank, with ascending ranks either from left to right or from right to left in your hand. The 4 suits can be in any order in your hand, and different suits can sorted in different directions.

Solution:

$$\Pr(\text{Sorted} \mid \text{At Least Two For Each Suit}) = \boxed{\frac{4! \cdot 2^4}{13!}}$$

For there are $4!$ ways to permute the suits and exactly two ways for each suit to be sorted.

1. Vampire City

For any individual x born in Transylvania with a vampire father, there is a 50% chance that x is a vampire, independently for each birth. These are the only conditions under which a new vampire can be created. 75% of the Transylvanian males are vampires. Suppose Igor, a man who has lived in Transylvania his whole life, has three children that are not vampires.

(a) What is the probability that Igor is a vampire?

Solution:

Let V be the event that Igor is a vampire. Let C_i be the event that Igor's i-th child is a vampire. Note that $\Pr(C_i \mid V) = 0.5$; if Igor is a vampire, each of his children has a 50% chance of being a vampire. Thus, $\Pr(C_i^C \mid V)$ is just $1 - \Pr(C_i \mid V)$ (imagine that we're restricting the sample space to outcomes where V happens), which is also 0.5. However, $\Pr(C_i \mid V^C) = 0$; if Igor is not a vampire, then it is impossible for his child to be a vampire. Thus, $\Pr(C_i^C \mid V^C) = 1 - 0 = 1$.

Apply Bayes' theorem and Law of Total Probability:

$$\begin{aligned}\Pr(V \mid C_1^C \cap C_2^C \cap C_3^C) &= \frac{\Pr(C_1^C \cap C_2^C \cap C_3^C \mid V)\Pr(V)}{\Pr(C_1^C \cap C_2^C \cap C_3^C)} \\ &= \frac{\Pr(C_1^C \cap C_2^C \cap C_3^C \mid V)\Pr(V)}{\Pr(C_1^C \cap C_2^C \cap C_3^C \mid V)\Pr(V) + \Pr(C_1^C \cap C_2^C \cap C_3^C \mid V^C)\Pr(V^C)}\end{aligned}$$

Apply independence:

$$\begin{aligned}\Pr(C_1^C \cap C_2^C \cap C_3^C \mid V) &= \Pr(C_1^C \mid V)\Pr(C_2^C \mid V)\Pr(C_3^C \mid V) = 0.5^3 \\ \Pr(C_1^C \cap C_2^C \cap C_3^C \mid V^C) &= \Pr(C_1^C \mid V^C)\Pr(C_2^C \mid V^C)\Pr(C_3^C \mid V^C) = 1^3\end{aligned}$$

We also know $\Pr(V) = 0.75$ and $\Pr(V^C) = 0.25$, so plug in the numbers:

$$\Pr(V \mid C_1^C \cap C_2^C \cap C_3^C) = \frac{0.5 \cdot 0.5 \cdot 0.5 \cdot 0.75}{0.5 \cdot 0.5 \cdot 0.5 \cdot 0.75 + 1 \cdot 1 \cdot 1 \cdot 0.25} = \boxed{\frac{3}{11}}$$

(b) If Igor has a fourth child, what is the probability that child will be a vampire?

Solution:

According to the problem, we need to restrict our sample space to outcomes where the first three children are not vampires (i.e. $C_1^C \cap C_2^C \cap C_3^C$). Let C^C be the event that the first three children are not vampires. In this case, $P(V|C^C) = \frac{3}{11}$ as derived in part (a). Apply the Law of Total Probability:

$$\begin{aligned}
\Pr(C_4 \mid C^C) &= \Pr(C_4 \mid V \cap C^C)\Pr(V \mid C^C) + \Pr(C_4 \mid V^C \cap C^C)\Pr(V^C \mid C^C) \\
&= 0.5 \cdot \frac{3}{11} + 0 \cdot (1 - \frac{3}{11}) \\
&= \boxed{\frac{3}{22}}
\end{aligned}$$

2. "Hey, let's poke your tires for fun!"

Bob is teaching Alice how to play his new favorite game. In each round, Bob shoots an arrow at the tires of Alice's car. He hits with probability p, independent of previous rounds. If he hits a tire, he gets 10 points. If he misses, he loses 5. Let X be Bob's score after n rounds.

(a) What is $\mathbb{E}[X]$?

Solution:

Let X_i be the number of points Bob gets for the i-th round, so $\mathbb{E}[X_i] = 10p + (-5)(1-p) = 15p - 5$.

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \boxed{15np - 5n}$$

(b) What is $\text{Var}(X)$?

Solution:

First calculate the variance of each X_i: $\text{Var}(X_i) = \mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2 = (10^2 p + (-5)^2(1-p)) - (15p-5)^2 = 225p(1-p)$.

$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \text{Var}(X_i) = \boxed{225np(1-p)}$$

3. $\frac{1}{\text{A Random Variable}}$

Let X be a continuous random variable with probability density function:

$$f_X(x) = \begin{cases} 2x & , \text{ if } 0 \le x \le 1 \\ 0 & , \text{ otherwise} \end{cases}$$

(a) Find $\mathbb{E}[\frac{1}{X}]$.

Solution:

$$\mathbb{E}\left[\frac{1}{X}\right] = \int_0^1 \frac{1}{x} \cdot 2x dx = \boxed{2}$$

(b) Compute $\Pr(X = 0.5)$.

Solution:

$$\text{Pr}(X = 0.5) = \boxed{0}$$

4. POWER SETS

Suppose A and B are random, independent, nonempty subsets of $\{1, 2, \ldots, n\}$, where each nonempty subset is equally likely to be chosen as A or B. What is $\text{Pr}(\max(A) = \max(B))$?

Solution:

We know $\text{Pr}(\max(A) = k) = \frac{2^{k-1}}{2^n-1}$, since for the max to be k, A needs to contain k and no element greater than k, and there are 2^{k-1} subsets of $\{1, 2, \ldots, k-1\}$. Now use the law of total probability:

$$\begin{aligned}
\text{Pr}(\max(A) = \max(B)) &= \sum_{k=1}^{n} \text{Pr}(\max(A) = \max(B) \mid \max(B) = k) \cdot \text{Pr}(\max(B) = k) \\
&= \sum_{k=1}^{n} \text{Pr}(\max(A) = k) \cdot \text{Pr}(\max(B) = k) \\
&= \sum_{k=1}^{n} \left(\frac{2^{k-1}}{2^n - 1}\right)^2 \\
&= \frac{1}{(2^n-1)^2} \sum_{k=1}^{n} 4^{k-1} \\
&= \frac{4^n - 1}{3(2^n-1)^2} = \frac{(2^n+1)(2^n-1)}{3(2^n-1)^2} = \boxed{\frac{2^n+1}{3(2^n-1)}}
\end{aligned}$$

5. Gumbel(μ, β)

Suppose $x_1, x_2, \ldots, x_n$ are independent samples from Gumbel(μ, β). A Gumbel distribution, or **Generalized Extreme Value distribution Type-I**, is used to model the distribution of the maximum (or the minimum) of a number of samples of various distributions. The probability density function for a typical Gumbel distribution is

$$f_X(x|\mu, \beta) = \frac{1}{\beta} \exp\left(-\frac{x-\mu}{\beta} - \exp\left(-\frac{x-\mu}{\beta}\right)\right)$$

where $\beta > 0$. Given that we know β, what is the maximum likelihood estimator for μ? Don't forget to prove that it is a maximum of the likelihood function.

Solution:

$$\begin{aligned}
L(x_1, x_2, ..., x_n|\mu, \beta) &= \prod_{i=1}^{n} \frac{1}{\beta} \exp\left(-\frac{x_i-\mu}{\beta} - \exp\left(-\frac{x_i-\mu}{\beta}\right)\right) \\
\ln L(x_1, x_2, ..., x_n|\mu, \beta) &= \sum_{i=1}^{n} \left(-\ln\beta - \frac{x_i-\mu}{\beta} - \exp\left(-\frac{x_i-\mu}{\beta}\right)\right) \\
\frac{\partial}{\partial\mu} \ln L(x_1, x_2, ..., x_n|\mu, \beta) &= \sum_{i=1}^{n} \left(\frac{1}{\beta} - \exp\left(-\frac{x_i-\mu}{\beta}\right) \cdot \frac{1}{\beta}\right) \\
&= \frac{n - \sum_{i=1}^{n} \exp\left(-\frac{x_i-\mu}{\beta}\right)}{\beta} \\
&= \frac{n - \exp(\frac{\mu}{\beta}) \sum_{i=1}^{n} \exp(-\frac{x_i}{\beta})}{\beta}
\end{aligned}$$

Set the first derivative to 0,

$$n = \exp\left(\frac{\hat{\mu}}{\beta}\right) \sum_{i=1}^{n} \exp\left(-\frac{x_i}{\beta}\right)$$

$$\ln n = \frac{\hat{\mu}}{\beta} + \ln\left(\sum_{i=1}^{n} \exp\left(-\frac{x_i}{\beta}\right)\right)$$

$$\hat{\mu} = \boxed{\beta\left(\ln n - \ln\left(\sum_{i=1}^{n} \exp\left(-\frac{x_i}{\beta}\right)\right)\right)}$$

To verify it's a global maximum,

$$\frac{\partial^2}{\partial \mu^2} \ln L(x_1, x_2, ..., x_n|\mu, \beta) = -\frac{\exp(\frac{\mu}{\beta}) \sum_{i=1}^{n} \exp(-\frac{x_i}{\beta})}{\beta^2} < 0$$

So $\ln L(x_1, x_2, ..., x_n|\mu, \beta)$ is concave downward everywhere with respect to μ.