CSE 312: Foundations of Computing II

Section 8: Continuous RV's and Central Limit Theorem

0. Transformations

Suppose $X \sim$ Uniform$(0,1)$ has the continuous uniform distribution on $(0,1)$. Let $Y = -\frac{1}{\lambda}\log X$ for some $\lambda > 0$.

(a) What is Ω_Y?

(b) First write down $F_X(x)$ for $x \in (0,1)$. Then, find $F_Y(y)$ on Ω_Y.

(c) Now find $f_Y(y)$ on Ω_Y. What distribution does Y have?

1. Poisson CLT

Suppose $X_1, \ldots, X_n$ are iid Poisson(λ) random variables, and let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$, the sample mean. How large should we choose n to be such that $\Pr\left(\frac{\lambda}{2} \le \overline{X}_n \le \frac{3\lambda}{2}\right) \ge 0.99$? Use the CLT and give an answer involving $\Phi^{-1}(\cdot)$. Then evaluate it exactly when $\lambda = 1/10$ using the Φ table on the last page.

2. Convolutions

Suppose $Z = X + Y$, where $X \perp Y$. Z is called the convolution of two random variables. If X, Y, Z are discrete,

$$p_Z(z) = \Pr(X + Y = z) = \sum_x \Pr(X = x \cap Y = z - x) = \sum_x p_X(x)\, p_Y(z - x)$$

If X, Y, Z are continuous,

$$F_Z(z) = \Pr(X + Y \le z) = \int_{-\infty}^{\infty} \Pr(Y \le z - X \mid X = x)\, f_X(x)dx = \int_{-\infty}^{\infty} F_Y(z - x) f_X(x)dx$$

Suppose $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$.

(a) Find an expression for $\Pr(X_1 < 2X_2)$ using a similar idea to convolution, in terms of $F_{X_1}, F_{X_2}, f_{X_1}, f_{X_2}$. (Your answer will be in the form of a single integral, and requires no calculations – do not evaluate it).

(b) Find s, where $\Phi(s) = \Pr(X_1 < 2X_2)$ using the fact that linear combinations of independent normal random variables are still normal.

3. Bad Computer

Each day, the probability your computer crashes is 10%, independent of every other day. Suppose we want to evaluate the computer's performance over the next 100 days.

(a) Let X be the number of crash-free days in the next 100 days. What distribution does X have? Identify $\mathbb{E}[X]$ and Var(X) as well. Write an exact (possibly unsimplified) expression for $\Pr(X \ge 87)$.

(b) Approximate the probability of at least 87 crash-free days out of the next 100 days using the Central Limit Theorem. Justify why we can use the CLT here.