CSE 312: Foundations of Computing II

Section 7: Continuous RV's Solutions

0. New PDF?

Alex came up with a function that he thinks could represent a probability density function. He defined the potential pdf for X as $f(x) = \frac{1}{1+x^2}$ defined on $[0, \infty)$. Is this a valid pdf? If not, find a constant c such that the pdf $f_X(x) = \frac{c}{1+x^2}$ is valid. Then find $\mathbb{E}[X]$. (Hints: $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$, $\tan \frac{\pi}{2} = \infty$, and $\tan 0 = 0$.)

Solution:

$$\int_0^\infty \frac{c}{1+x^2} dx = c \tan^{-1} x \,|_0^\infty = c\left(\frac{\pi}{2} - 0\right) = 1$$

so $c = 2/\pi$.

$$\mathbb{E}[X] = \int_0^\infty \frac{cx}{1+x^2} dx = \frac{2}{\pi} \int_0^\infty \frac{x}{1+x^2} dx = \frac{1}{\pi} \ln(1+x^2) \,|_0^\infty = \infty$$

1. Uniform2

Alex decided he wanted to create a "new" type of distribution that will be famous, but he needs some help. He knows he wants it to be continuous and have uniform density, but he needs help working out some of the details. We'll denote a random variable X having the "Uniform-2" distribution as $X \sim$ Uniform2(a, b, c, d), where $a < b < c < d$. We want the density to be non-zero in $[a, b]$ and $[c, d]$, and zero everywhere else. Anywhere the density is non-zero, it must be equal to the same constant.

(a) Find the probability density function, $f_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty, \infty)$. (Hint: use a piecewise definition).

Solution:

$$f_X(x) = \begin{cases} \frac{1}{(b-a)+(d-c)}, & x \in [a,b] \cup [c,d] \\ 0, & \text{otherwise} \end{cases}$$

(b) Find the cumulative distribution function, $F_X(x)$. Be sure to specify the values it takes on for every point in $(-\infty, \infty)$. (Hint: use a piecewise definition).

Solution:

$$F_X(x) = \begin{cases} 0, & x \in (-\infty, a) \\ \frac{(x-a)}{(b-a)+(d-c)}, & x \in [a, b) \\ \frac{(b-a)}{(b-a)+(d-c)}, & x \in [b, c) \\ \frac{(b-a)+(x-c)}{(b-a)+(d-c)}, & x \in [c, d) \\ 1, & x \in [d, \infty) \end{cases}$$

2. Continuous Law of Total Probability?

In this exercise, we will extend the law of total probability to the continuous case.

(a) Suppose we flip a coin with probability U of heads, where U is equally likely to be one of $\Omega_U = \{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$ (notice this set has size $n+1$). Let H be the event that the coin comes up heads. What is Pr(H)?

Solution:

We can use the law of total probability, conditioning on $U = \frac{k}{n}$ for $k = 0, ..., n$.

$$\text{Pr}(H) = \sum_{k=0}^{n} \text{Pr}\left(H \mid U = \frac{k}{n}\right) \text{Pr}\left(U = \frac{k}{n}\right) = \sum_{k=0}^{n} \frac{k}{n} \cdot \frac{1}{n+1} = \frac{1}{n(n+1)} \sum_{k=0}^{n} k = \frac{1}{n(n+1)} \frac{n(n+1)}{2} = \frac{1}{2}$$

(b) Now suppose $U \sim$ Uniform(0,1) has the *continuous* uniform distribution over the interval $[0, 1]$. Extend the law of total probability to work for this continuous case. (Hint: you may have an integral in your answer instead of a sum).

Solution:

$$\text{Pr}(H) = \int_0^1 \text{Pr}(H \mid U = u)\, f_U(u) du = \int_0^1 u \cdot 1 du = \frac{1}{2}[u^2]_0^1 = \frac{1}{2}$$

(c) Let's generalize the previous result we just used. Suppose E is an event, and X is a continuous random variable with density function $f_X(x)$. Write an expression for $\text{Pr}(E)$, conditioning on X.

Solution:

$$\text{Pr}(E) = \int_{-\infty}^{\infty} \text{Pr}(E \mid X = x)\, f_X(x) dx$$