CSE 312: Foundations of Computing II

Section 6: Midterm review Solutions

0. Mutually Independent

Let A and B be events in the same sample space that each have nonzero probability. For each of the following statements, state whether it is always true, always false, or it depends on information not given.

(a) If A and B are mutually exclusive, then they are independent.

Solution:

False

(b) If A and B are independent, then they are mutually exclusive.

Solution:

False

(c) If Pr(A) = Pr(B) = 0.75, then A and B are mutually exclusive.

Solution:

False

(d) If Pr(A) = Pr(B) = 0.75, then A and B are independent.

Solution:

Depends whether Pr$(A \cap B) = 9/16$

1. A Pond of Random Variable Fish

Suppose I am fishing in a pond with B blue fish, R red fish, and G green fish, where $B + R + G = N$. For each of the following scenarios, identify the most appropriate distribution (with parameter(s)):

(a) how many of the next 10 fish I catch are blue, if I catch and release

Solution:

$$\text{Binomial}\left(10, \frac{B}{N}\right)$$

(b) how many fish I had to catch until my first green fish, if I catch and release

Solution:

$$\text{Geometric}\left(\frac{G}{N}\right)$$

(c) how many red fish I catch in the next five minutes, if I catch on average r red fish per minute

Solution:

$$\text{Poisson}(5r)$$

(d) whether or not my next fish is blue

Solution:

$$\text{Bernoulli}\left(\frac{B}{N}\right)$$

2. Plusses and a's

Consider the following inequality: $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \leq 70$. A solution to this inequality over the nonnegative integers is a choice of a nonnegative integer for each of the 6 variables $a_1, a_2, a_3, a_4, a_5, a_6$ that satisfies the inequality. To be different, two solutions have to differ on the value assigned to some a_i. How many different solutions are there to the inequality? (This problem is similar to a problem from HW1, except that "$=$" has been replaced by "$\leq$".)

Solution:

This is equivalent to asking how many different solutions are there to the equation $a_1+a_2+a_3+a_4+a_5+a_6+a_7 = 70$. The answer is $\binom{76}{6}$, using the method from HW1 (stars and bars).

3. The Elevator Problem

A building has n floors numbered $1, 2, \cdots, n$, plus a ground floor G. At the ground floor, m people get on the elevator together, and uniformly, each gets off randomly at one of the n floors (independently of everybody else). What is the expected number of floors the elevators stops at (not counting the ground floor)?

Solution:

For $i = 1$ to n, let $X_i = 1$ if the elevator stops at the i-th floor, and let $X_i = 0$ otherwise. Then N, the number of floors stopped at, is given by

$$N = X_1 + X_2 + \cdots + X_n.$$

Thus by the linearity of expectation, we have

$$\mathbb{E}[N] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n].$$

The probability that no one stops at floor i is $\left(\frac{n-1}{n}\right)^m$.
Thus the probability that $X_i = 1$ is $1 - \left(\frac{n-1}{n}\right)^m$. This is also the expectation of X_i.
For $\mathbb{E}[N]$, add up over all i, that is, multiply by n.

4. Blood Sugar and Diabetes

You are trying to diagnose the probability that a patient with a positive blood sugar test result has diabetes, even though she is in a low risk group. The probability of a woman in this group having diabetes is 0.8%. 90% of women with diabetes will test positive in the blood sugar test. 7% of women without diabetes will test positive in the blood sugar test. Your patient tests positive in the blood sugar test. What is the probability that she has diabetes?

Solution:

Let D be the event that she has diabetes and $+$ be the event of a positive test.

$$\Pr(D \mid +) = \frac{\Pr(+ \mid D)\Pr(D)}{\Pr(+ \mid D)\Pr(D) + \Pr(+ \mid \overline{D})\Pr(\overline{D})} = \frac{0.9 \times 0.008}{0.9 \times 0.008 + 0.07 \times 0.992} \approx 0.09$$

Notice that the posterior probability 0.09 of diabetes is approximately 10 times as great as the prior probability 0.008 of diabetes, but still small.

5. Hey Charlie

A very long multiple choice exam has 4 choices for each question. Charlie has studied enough so that he knows the correct answer for 1/2 of the questions; for an additional 1/4 of the questions he can eliminate one choice and chooses randomly and uniformly among the other three, and for the remaining 1/4 of the questions he chooses randomly and uniformly among all four answers.

As the teacher, you want to determine how many answers the student actually knows. For a randomly chosen question, if Charlie answers it correctly, what is the probability he knew the answer?

Solution:

Let C be the event that Charlie has the correct answer and K be the event that Charlie knew the answer. Then

$$\begin{aligned} \Pr(C) &= \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{1}{4} = \frac{31}{48} \\ \Pr(K \mid C) &= \frac{\Pr(C \mid K) \Pr(K)}{\Pr(C)} = \frac{1 \cdot \frac{1}{2}}{31/48} = \frac{24}{31} \end{aligned}$$

6. Martian Meteors

Suppose the number of meteors hitting Mars in any time interval is a Poisson random variable. Mars gets on average 5 strikes per year.

(a) What is the probability that there will be no strikes in an interval of 2 years?

Solution:

$X \sim \text{Poisson}(2 * 5)$
$\Pr(X = k) = e^{-10} * \frac{10^k}{k!}$
$\Pr(X = 0) = e^{-10} * \frac{10^0}{0!} = e^{-10}$

(b) What is the probability that there is at least one hit in an interval of one year?

Solution:

$\lambda = 5$, $X \sim \text{Poisson}(5)$, $\Pr(X \geq 1) = 1 - \Pr(X = 0) = 1 - \frac{e^{-5}5^0}{0!} = 1 - e^{-5}$