CSE 312: Foundations of Computing II

Section 6: Midterm Review Solutions

0. Randomized Algorithms

Suppose we have a randomized algorithm $\mathcal{R}$ which takes input, and does one of three things: returns the correct answer, times out, or returns an incorrect answer. Call these possible actions, C, T, and I respectively. Suppose for each run of $\mathcal{R}$, independently, $\Pr(C) = 0.51$, $\Pr(T) = 0.4$, and $\Pr(I) = 0.09$. Each part of this question is independent of the others; only the information above should be considered when answering each part.

(a) Suppose we run this algorithm until it returns the correct answer, but not running it more than m times. How many different output sequences are possible?

Solution:

For $k = 1, ..., m$, the number of possible outcomes if we get the correct answer on run k is 2^{k-1}. Hence, we have $\sum_{k=1}^{m} 2^{k-1} = 2^m - 1$. If we don't get the correct answer, there are 2^m ways. So the total is $2^m - 1 + 2^m = 2^{m+1} - 1$.

(b) Let X be the number of runs until the tenth time out. What is the codomain Ω_X, and what is $p_X(k) = \Pr(X = k)$?

Solution:

$\Omega_X = \{10, 11, 12, ...\}$ since it takes at least ten runs to get ten time outs. For the k^{th} run to be the tenth timeout, anywhere in the first $k - 1$, we need 9 timeouts, and $k - 9$ non-timeouts. Finally the last run must be a timeout, so the PMF is $p_X(k) = \Pr(X = k) = \binom{k-1}{9}(0.4)^9(0.6)^{k-9}(0.4)$.

(c) Let X be the rv from the previous part. What is $\mathbb{E}[X]$?

Solution:

For $i = 1, ..., 10$, let X_i be the number of runs until the i^{th} timeout, from the previous timeout, so that $X = \sum_{i=1}^{10} X_i$. Then, $X_i \sim \text{Geometric}(0.4)$, so $\mathbb{E}[X_i] = \frac{1}{0.4} = 2.5$. Hence $\mathbb{E}[X] = \sum_{i=1}^{10} \mathbb{E}[X_i] = 10 \cdot 2.5 = 25$.

(d) Suppose we only have enough computational resources to run this algorithm at most m times. What is the probability that $\mathcal{R}$ returns a correct answer at least once within m runs?

Solution:

Let E be the event desired. Then $\bar{E}$ is the event that we never get a correct answer in m runs. $\Pr(E) = 1 - \Pr(\bar{E}) = 1 - (1 - 0.51)^m$.

(e) Let's say we run $\mathcal{R}$ 100 times. What is the probability we get exactly 23 correct answers, 66 timeouts, and 11 incorrect answers?

Solution:

There are $\binom{100}{23,66,11} = \binom{100}{23}\binom{77}{66}$ ways to choose the locations of each outcome. Hence the probability is $\binom{100}{23,66,11}(0.51)^{23}(0.4)^{66}(0.09)^{11}$.

(f) What is the probability we get an incorrect answer before a correct answer while running $\mathcal{R}$ repeatedly?

Solution:

We can ignore any timeouts, since they don't affect this problem. So the updated probabilities are $\Pr(C \mid \bar{T}) = \frac{\Pr(C)}{\Pr(C)+\Pr(I)} = \frac{0.51}{0.51+0.09} = \frac{51}{60}$. So $\Pr(I \mid \bar{T}) = \frac{9}{60}$. Hence the desired probability is $\frac{9}{60}$.

(g) For this last part, assume $\Pr(C) = \Pr(T) = \Pr(I) = \frac{1}{3}$. Suppose we run $\mathcal{R}$ exactly $2m+1$ times. We can treat its output $\mathcal{O}$ as a string in $\{C, T, I\}^{2m+1}$. What is the probability that $\mathcal{O}$ is a palindrome that starts or ends with C, and has no two consecutive letters?

Solution:

First realize we can only choose the first $m+1$, since the last m will have to be the reverse of the first m. There is only one choice for the first letter, C. Subsequently, there are only 2 choices for each of the next m letters. So the number of palindromes satisfying the above is 2^m, out of the 3^{2m+1} equally likely strings. Hence the desired probability is $\frac{2^m}{3^{2m+1}}$.

1. Love Triangles

Suppose we have n people with amnesia in a prison. The prison guard tells each pair of two people whether or not they are in love randomly, with probability 0.7 of telling them they are. These people, having amnesia, are especially suggestive and believe the prison guard. What is the expected number of love triangles? (A love triangle is a group of 3 people who all love each other). If more than 3 people all love each other, count each group of 3 separately. E.g., if A,B,C, and D all love each other, count four triangles for each group of three.

Solution:

We'll use linearity of expectation. There are $m = \binom{n}{3}$ possible love triangles. Let $X_1, ..., X_m$ be indicator rvs, where X_i is 1 if the i^{th} group forms a love triangle, and 0 otherwise. Then, $\mathbb{E}[X_i] = \Pr(\text{all three love each other}) = 0.7^3$. Hence, $\mathbb{E}[X] = \sum_{i=1}^{m} \mathbb{E}[X_i] = \binom{n}{3}0.7^3$.

2. Space Shuttles

The space shuttle has 6 O-rings: these were involved in the Challenger disaster. When the space shuttle is launched, each O-ring has a probability of failure of 0.0137, independent of whether other O-rings fail.

(a) What is the probability that, during 23 launches, no O-ring will fail, but that at least one O-ring will fail during the 24th launch?

Solution:

The probability that no O-ring fails on a single launch is $(1-0.0137)^6 \approx 0.921$. The probability that this happen for 23 launches and doesn't happen on the 24th launch is $0.921^{23}(1-0.921) \approx 0.0118$.

(b) What is the probability that no O-ring fails during 24 launches?

Solution:

$0.921^{24} \approx 0.137$