CSE 312: Foundations of Computing II

QuickCheck: Midterm Review Solutions (due Thursday, May 3)

0. I Cast Arcane Missiles!

Suppose you and your friend are dueling each other. He just summoned a minion with two health, and you have three shots of arcane missiles each dealing one damage. Knowing that your friend has 10 health, you decided to use all of the missiles and see if you can defeat his minion. Arcane missiles target randomly between that minion and your friend, but since you don't want to waste any firepower, they will not hit targets that have zero or less health. Let X be a random variable that denotes the number of missiles that hit your friend.

(a) Find the codomain of X and $p_X(k)$, the probability mass function for X.

Solution:

The codomain of X is $\{1, 2, 3\}$.

Let M denotes targeting minion, and F denotes targeting your friend, there are seven possible cases. Six of them, $FFF, FFM, FMF, FMM, MFF, MFM$, with probability $\frac{1}{8}$, and MMF with probability $\frac{1}{4}$ because the last missile must target your friend. Sums them up, we get:

$$p_X(k) = \begin{cases} \frac{4}{8} & k = 1 \\ \frac{3}{8} & k = 2 \\ \frac{1}{8} & k = 3 \end{cases}$$

(b) Find $\mathbb{E}[X]$ by definition of expectation.

Solution:

$$\mathbb{E}[X] = \frac{4}{8} \cdot 1 + \frac{3}{8} \cdot 2 + \frac{1}{8} \cdot 3 = \frac{13}{8}$$

(c) Sharpnel insisted on using linearity of expectation to find $\mathbb{E}[X]$. He wrote the following:

Let X_1, X_2, X_3 be random variables that denote the outcome for each arcane missile: 1 if it hit your friend, and 0 if it hit the minion.

$$\mathbb{E}[X_1] = \mathbb{E}[X_2] = \mathbb{E}[X_3] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2} \quad (1)$$

$$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2 + X_3] \quad (2)$$

$$= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] \quad (3)$$

$$= \frac{3}{2} \quad (4)$$

"And I just broke the Linearity of Expectation!" He yelled in happiness, trying to utilize this incredible result. If he is correct (which shouldn't be), say so, else briefly explain his error.

Solution:

$\mathbb{E}[X_3] \neq \frac{1}{2}$, because the case MMM has probability 0 and MMF has probability $\frac{1}{4}$. Among all possibilities, there are three cases involving the third missile targeting that minion, each with probability $\frac{1}{8}$, and four cases for the third missile to target your friend, with three of them have $\frac{1}{8}$ and one of them has $\frac{1}{4}$. Therefore,

$$\mathbb{E}[X_3] = \frac{5}{8} \cdot 1 + \frac{3}{8} \cdot 0 = \frac{5}{8}$$

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] = \frac{1}{2} + \frac{1}{2} + \frac{5}{8} = \frac{13}{8}$$