CSE 312: Foundations of Computing II

Section 5: Variance, Important Discrete Distributions Solutions

0. Best Coach Ever!!

You are a hardworking boxer. Your coach tells you that the probability of your winning a boxing match is 0.2 independently of every other match.

(a) How many matches do you expect to fight until you win 10 times ?

Solution:

The number of matches you have to fight until you win 10 times can be modeled by $\sum_{i=1}^{10} X_i$ where $X_i \sim$ Geometric(0.2) is the number of matches you have to fight to win the i^{th} time. Recall $\mathbb{E}[X_i] = \frac{1}{0.2} = 5$. $\mathbb{E}\left[\sum_{i=1}^{10} X_i\right] = \sum_{i=1}^{10} \mathbb{E}[X_i] = \sum_{i}^{10} \frac{1}{0.2} = 10 \cdot 5 = 50.$

(b) You only get to play 12 matches every year. To win a spot in the Annual Boxing Championship, a boxer needs to win at least 10 matches in a year. What is the probability that you will go to the Championship this year?

Solution:

You can go to the championship if you win more than or equal to 10 times this year. Let Y be the number of matches you win out of the 12 matches. Note that $Y \sim$ Binomial$(12, 0.2)$. We are interested in

$$\text{Pr}(Y = 10) + \text{Pr}(Y = 11) + \text{Pr}(Y = 12) = \sum_{i=10}^{12} \binom{12}{i} 0.2^i (1 - 0.2)^{12-i}$$

(c) Let p be your answer to part (b). How many times can you expect to go to the Championship in your 20 year career?

Solution:

The number of times you go to the championship can be modeled by $Y \sim$ Binomial$(20, p)$. So, $E[Y] = 20 \cdot p$.

1. Variance of a Product

Let X, Y, Z be independent random variables with means μ_X, μ_Y, μ_Z and variances $\sigma_X^2, \sigma_Y^2, \sigma_Z^2$, respectively. Find Var$(XY - Z)$.

Solution:

First notice that $\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \implies \mathbb{E}[X^2] = \text{Var}(X) + \mathbb{E}[X]^2 = \sigma_X^2 + \mu_X^2$, and same for Y.

$$\begin{aligned} \text{Var}(XY) &= \mathbb{E}[X^2Y^2] - \mathbb{E}[XY]^2 \text{ (by theorem in class)} \\ &= \mathbb{E}[X^2]\,\mathbb{E}[Y^2] - \mathbb{E}[X]^2\,\mathbb{E}[Y]^2 \text{ (by independence)} \\ &= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2 \end{aligned}$$

By independence,

$$\begin{aligned} \text{Var}(XY - Z) &= \text{Var}(XY) + \text{Var}(Z) \\ &= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2\mu_Y^2 + \sigma_Z^2 \end{aligned}$$

2. Deception is my last name

Suppose we roll two fair 5-sided dice independently. Let X be the value of the first die, Y be the value of the second die, $Z = X + Y$ be their sum, $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$.

(a) Find $p_U(u)$.

Solution:

$$p_U(u) = \begin{cases} \frac{9}{25} & u = 1 \\ \frac{7}{25} & u = 2 \\ \frac{5}{25} & u = 3 \\ \frac{3}{25} & u = 4 \\ \frac{1}{25} & u = 5 \end{cases}$$

(b) Find $\mathbb{E}[U]$.

Solution:

Use definition of Expectation: $\mathbb{E}[U] = 2.2$

(c) Find $\mathbb{E}[Z]$.

Solution:

By linearity of expectation, $\mathbb{E}[Z] = \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 3 + 3 = 6$.

(d) Find $\mathbb{E}[UV]$. (Note that U and V are **not** independent. Why not?).

Solution:

$\mathbb{E}[UV] = \mathbb{E}[XY] = \mathbb{E}[X]\, E[Y] = 3^2 = 9$ Since $UV = XY$ (why?), and then X, Y are independent.

(e) Find Var$(U + V)$. You may use the fact that, if $W \sim$ Uniform(a, b) (meaning equally likely to be any integer $\{a, a + 1, ..., b\}$), then Var$(W) = \frac{(b-a)(b-a+2)}{12}$.

Solution:

Since $X, Y \sim$ Uniform$(1, 5)$, Var$(X) =$ Var$(Y) = \frac{(5-1)\cdot(5-1+2)}{12} = 2$

$$\text{Var}(U + V) = \text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) = 2 + 2 = 4$$

because X and Y are independent and $U + V = X + Y$ (why?).

3. True or False?

Identify the following statements as true or false (true means always true). Justify your answer.

(a) For any random variable X, we have $\mathbb{E}\left[X^2\right] \geq \mathbb{E}[X]^2$.

Solution:

True, since $0 \leq$ Var$(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$.

(b) Let X, Y be random variables. Then, X and Y are independent if and only if $\mathbb{E}[XY] = \mathbb{E}[X]\, \mathbb{E}[Y]$.

Solution:

False. The forward implication is true, but the reverse is not. For example, if $X \sim$ Uniform$(-1, 1)$ (equally likely to be in $\{-1, 0, 1\}$), and $Y = X^2$, we have $\mathbb{E}[X] = 0$, so $\mathbb{E}[X]\,\mathbb{E}[Y] = 0$. However, since $X = X^3$ (why?), $\mathbb{E}[XY] = \mathbb{E}\big[XX^2\big] = \mathbb{E}\big[X^3\big] = \mathbb{E}[X] = 0$, we have that $\mathbb{E}[X]\,\mathbb{E}[Y] = 0 = \mathbb{E}[XY]$. However, X and Y are not independent; indeed, $\Pr(Y = 0 \mid X = 0) = 1 \neq \frac{1}{3} = \Pr(Y = 0)$.

(c) Let $X \sim$ Binomial(n, p) and $Y \sim$ Binomial(m, p) be independent. Then, $X + Y \sim$ Binomial$(n + m, p)$.

Solution:

True. X is the sum of n independent Bernoulli trials, and Y is the sum of m. So $X + Y$ is the sum of $n + m$ independent Bernoulli trials, so $X + Y \sim$ Binomial$(n + m, p)$.

(d) Let $X_1, ..., X_{n+1}$ be independent Bernoulli(p) random variables. Then, $\mathbb{E}[\sum_{i=1}^{n} X_i X_{i+1}] = np^2$.

Solution:

True. Notice that $X_i X_{i+1}$ is also Bernoulli (only takes on 0 and 1), but is 1 iff both are 1, so $X_i X_{i+1} \sim$ Bernoulli(p^2). The statement holds by linearity, since $\mathbb{E}[X_i X_{i+1}] = p^2$.

(e) Let $X_1, ..., X_{n+1}$ be independent Bernoulli(p) random variables. Then, $Y = \sum_{i=1}^{n} X_i X_{i+1} \sim$ Binomial(n, p^2).

Solution:

False. They are all Bernoulli p^2 as determined in the previous part, but they are not independent. Indeed, $\Pr(X_1 X_2 = 1 \mid X_2 X_3 = 1) = \Pr(X_1 = 1) = p \neq p^2 = \Pr(X_1 X_2 = 1)$.

(f) If $X \sim$ Bernoulli(p), then $nX \sim$ Binomial(n, p).

Solution:

False. The range of X is $\{0, 1\}$, so the range of nX is $\{0, n\}$. nX cannot be Bin(n, p), otherwise its range would be $\{0, 1, ..., n\}$.

(g) If $X \sim$ Binomial(n, p), then $\frac{X}{n} \sim$ Bernoulli(p).

Solution:

False. Again, the range of X is $\{0, 1, ..., n\}$, so the range of $\frac{X}{n}$ is $\{0, \frac{1}{n}, \frac{2}{n}, ..., 1\}$. Hence it cannot be Ber$(p)$, otherwise its range would be $\{0, 1\}$.

(h) For any two independent random variables X, Y, we have Var$(X - Y) =$ Var$(X) -$ Var(Y).

Solution:

False. Var$(X - Y) =$ Var$(X + (-Y)) =$ Var$(X) + (-1)^2$Var$(Y) =$ Var$(X) +$ Var(Y).