CSE 312: Foundations of Computing II

Section 5: Variance, Important Discrete Distributions Solutions

0. The Enemy of my Enemy is my Friend

Suppose we have n people with amnesia in a prison. The prison guard tells each pair of two people whether or not they are enemies randomly, with probability 0.8 of telling them they are. These people, having amnesia, are especially suggestive and believe the prison guard.

(a) What is the expected number of pairs of enemies?

Solution:

We'll use linearity of expectation. There are $m = \binom{n}{2}$ possible pairs. Let $X_1, ..., X_m$ be indicator rvs, where X_i is 1 if the i^{th} pair are enemies, and 0 otherwise. Then, $\mathbb{E}[X_i] = \Pr(X_i = 1) = 0.8$. Hence, $\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = 0.8 \cdot \binom{n}{2}$.

(b) What is the variance of number of pairs of enemies?

Solution:

We'll use linearity of variance. This requires independence of the rv's, which we do indeed have. There are $m = \binom{n}{2}$ possible pairs. Let $X_1, ..., X_m$ be indicator rvs, where X_i is 1 if the i^{th} pair are enemies, and 0 otherwise. Then, $\text{Var}(X_i) = (0.8)(1 - 0.8) = 0.16$. Hence, $\text{Var}(X) = \sum_{i=1}^{n} \text{Var}(X_i) = 0.16 \cdot \binom{n}{2}$.

(c) If A and B are both enemies to C, then A and B become friends. A and B can become friends even if they are enemies (they will be "frenemies"). What is the expected number of friendships that are created as a result?

Solution:

We'll use linearity of expectation. There are $m = \binom{n}{2}$ possible pairs. Let $X_1, ..., X_m$ be indicator rvs, where X_i is 1 if the i^{th} pair are friends, and 0 otherwise. Then,

$$\mathbb{E}[X_i] = \Pr(X_i = 1) = \Pr(\text{there exists another person which they both hate})$$

$$= 1 - \Pr(\text{for every other person, they don't both hate that person}) = 1 - (1 - 0.8^2)^{n-2}$$

. Hence, $\mathbb{E}[X] = \sum_{i=1}^{m} \mathbb{E}[X_i] = \binom{n}{2}(1 - (1 - 0.8^2)^{n-2})$.

(d) What is the expected number of pairs of "frenemies"? We say a pair of people are frenemies if and only if they are enemies and friends.

Solution:

We'll use linearity of expectation. This is the same as the previous part, except $\Pr(X_i = 1) = 0.8(1 - (1 - 0.8^2)^{n-2})$, since the pair also has to be enemies. So the expected number of frenemies is $0.8 \cdot \binom{n}{2}(1 - (1 - 0.8^2)^{n-2})$

(e) What is the expected number of pairs of "true" enemies? We say a pair of people are true enemies if and only if they are enemies, but not friends.

Solution:

We'll use linearity of expectation. Let Z be the number of "true" enemies. Let X be the number of enemies, and Y be the number of frenemies. Then the $Z = X - Y$, and $\mathbb{E}[Z] = \mathbb{E}[X] - \mathbb{E}[Y] = 0.8 \cdot \binom{n}{2} - 0.8 \cdot \binom{n}{2}(1 - (1 - 0.8^2)^{n-2}) = 0.8 \cdot \binom{n}{2}((1 - 0.8^2)^{n-2})$