CSE 312: Foundations of Computing II

Section 4: Random Variables, Linearity of Expectation

0. Identify that Range!

Identify the support/codomain Ω_X of the random variable X, if X is...

(a) The sum of two rolls of a six-sided die.

(b) The number of lottery tickets I buy until I win it.

(c) The number of heads in n flips of a coin with $0 < \Pr(\text{head}) < 1$.

(d) The number of heads in n flips of a coin with $\Pr(\text{head}) = 1$.

(e) The time I wait at the bus stop for the next bus.

(f) The number of people born in the next year.

1. Kit Kats Again

Suppose we have N candies in a jar, K of which are kit kats. Suppose we draw (without replacement) until we have (exactly) k kit kats, $k \leq K \leq N$. Let X be the number of draws until the k^{th} kit kat. What is Ω_X, the codomain of X? What is $p_X(n) = \Pr(X = n)$? (We say X is a "negative hypergeometric" random variable).

2. Frogger

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_1, to the left with probability p_2, and doesn't move with probability p_3, where $p_1 + p_2 + p_3 = 1$. After 2 seconds, let X be the location of the frog.

(a) Determine the codomain Ω_X of X, and $p_X(k)$ the probability mass function for X.

(b) Compute $\mathbb{E}[X]$ from the definition.

(c) Compute $\mathbb{E}[X]$ again, but using linearity of expectation. (You will find that $\mathbb{E}[X] = 2(p_1 - p_2)$, which is actually equivalent to the previous answer after simplification).

3. Frogger Again!

The purpose of this problem is to show that in general, $\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$. Recall $\mathbb{E}[g(X)] = \sum_{x \in \Omega_X} g(x) p_X(x)$. Consider the frog in the previous problem. We computed that $\mathbb{E}[X] = 2(p_1 - p_2)$.

(a) Let $Y = g(X) = |X|$. Interpret what Y represents in English, and identity the codomain Ω_Y.

(b) Find the probability mass function for Y, $p_Y(k) = \Pr(Y = k)$. Then determine $\mathbb{E}[Y] = \sum_{y \in \Omega_Y} y p_Y(y)$. Verify that it equals the previous formula $\mathbb{E}[Y] = \sum_{x \in \Omega_X} g(x) p_X(x) = \sum_{x \in \Omega_X} |x| p_X(x)$.

(c) Let's give some concrete values: let $p_1 = p_2 = \frac{1}{2}$, and $p_3 = 0$ (so the frog just moves left and right with equal probability, and can't stay still). Demonstrate $\mathbb{E}[\,|X|] \neq |\mathbb{E}[X]\,|$ (Recall $Y = |X|$, so $\mathbb{E}[\,|X|] = \mathbb{E}[Y]$).

4. Hungry Washing Machine

You have 10 pairs of socks (so 20 socks in total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks chosen at random. Every subset of 4 socks is equally probable to be the subset that gets eaten. Let X be the number of complete pairs of socks that you have left.

(a) What is the codomain of X, Ω_X (the set of possible values it can take on)? What is the probability mass function of X?

(b) Find $\mathbb{E}[X]$ from the definition of expectation.

(c) Find $\mathbb{E}[X]$ using linearity of expectation.

(d) Which way was easier? Doing both (a) and (b), or just (c)?