CSE 312: Foundations of Computing II

Section 4: Random Variables, Linearity of Expectation Solutions

0. Identify that Range!

Identify the support/codomain Ω_X of the random variable X, if X is...

(a) The sum of two rolls of a six-sided die.

Solution:

$\Omega_X = \{2, 3, ..., 12\}$

(b) The number of lottery tickets I buy until I win it.

Solution:

$\Omega_X = \{1, 2, ...\} = \mathbb{N}$

(c) The number of heads in n flips of a coin with $0 < \text{Pr}(\text{head}) < 1$.

Solution:

$\Omega_X = \{0, 1, ..., n\}$

(d) The number of heads in n flips of a coin with $\text{Pr}(\text{head}) = 1$.

Solution:

$\Omega_X = \{n\}$

(e) The time I wait at the bus stop for the next bus.

Solution:

$\Omega_X = [0, \infty)$

(f) The number of people born in the next year.

Solution:

$\Omega_X = \{0, 1, 2, ...\}$

1. Kit Kats Again

Suppose we have N candies in a jar, K of which are kit kats. Suppose we draw (without replacement) until we have (exactly) k kit kats, $k \leq K \leq N$. Let X be the number of draws until the k^{th} kit kat. What is Ω_X, the codomain of X? What is $p_X(n) = \text{Pr}(X = n)$? (We say X is a "negative hypergeometric" random variable).

Solution:

$$\Omega_X = \{k, k+1, \ldots N - K + k\}$$

$$p_X(n) = \text{Pr}(X = n) = \frac{\binom{K}{k-1}\binom{N-K}{n-k}}{\binom{N}{n-1}} \; \frac{K - (k-1)}{N - (n-1)}, \; n \in \Omega_X$$

2. Frogger

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_1, to the left with probability p_2, and doesn't move with probability p_3, where $p_1+p_2+p_3 = 1$. After 2 seconds, let X be the location of the frog.

(a) Determine the codomain Ω_X of X, and $p_X(k)$ the probability mass function for X.

Solution:

Let L be a left step, R be a right step, and N be no step.

The codomain of X is $\{-2,-1,0,1,2\}$. We can compute $p_X(-2) = \Pr(X = -2) = \Pr(LL) = p_2^2$, $p_X(-1) = \Pr(X = -1) = \Pr(LN \cup NL) = 2p_2p_3$, and $p_X(0) = \Pr(X = 0) = \Pr(NN \cup LR \cup RL) = p_3^2 + 2p_1p_2$. Similarly for $p_X(1)$ and $p_X(2)$.

$$p_X(k) = \begin{cases} p_2^2 & k = -2 \\ 2p_2p_3 & k = -1 \\ p_3^2 + 2p_1p_2 & k = 0 \\ 2p_1p_3 & k = 1 \\ p_1^2 & k = 2 \end{cases}$$

(b) Compute $\mathbb{E}[X]$ from the definition.

Solution:

$$\mathbb{E}[X] = (-2)(p_2^2) + (-1)(2p_2p_3) + (0)(p_3^2 + 2p_1p_2) + (1)(2p_1p_3) + (2)(p_1^2) = 2(p_1 - p_2)$$

(c) Compute $\mathbb{E}[X]$ again, but using linearity of expectation. (You will find that $\mathbb{E}[X] = 2(p_1 - p_2)$, which is actually equivalent to the previous answer after simplification).

Solution:

Let Y be the amount you moved on the first step (either $-1, 0, 1$), and Z the amount you moved on the second step. Then, $\mathbb{E}[Y] = \mathbb{E}[Z] = (1)(p_1) + (0)(p_3) + (-1)(p_2) = p_1 - p_2$.

Then $X = Y + Z$ and $\mathbb{E}[X] = \mathbb{E}[Y + Z] = \mathbb{E}[Y] + \mathbb{E}[Z] = 2(p_1 - p_2)$

3. Frogger Again!

The purpose of this problem is to show that in general, $\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$. Recall $\mathbb{E}[g(X)] = \sum_{x\in\Omega_X} g(x)p_X(x)$. Consider the frog in the previous problem. We computed that $\mathbb{E}[X] = 2(p_1 - p_2)$.

(a) Let $Y = g(X) = |X|$. Interpret what Y represents in English, and identity the codomain Ω_Y.

Solution:

Y is the absolute value of X, which is just the absolute distance from the origin, so $\Omega_Y = \{0, 1, 2\}$.

(b) Find the probability mass function for Y, $p_Y(k) = \Pr(Y = k)$. Then determine $\mathbb{E}[Y] = \sum_{y\in\Omega_Y} yp_Y(y)$. Verify that it equals the previous formula $\mathbb{E}[Y] = \sum_{x\in\Omega_X} g(x)p_X(x) = \sum_{x\in\Omega_X} |x|p_X(x)$.

Solution:

$$p_Y(k) = \begin{cases} p_3^2 + 2p_1p_2 & k = 0 \\ 2p_1p_3 + 2p_2p_3 & k = 1 \\ p_1^2 + p_2^2 & k = 2 \end{cases}$$

$$\mathbb{E}[Y] = 0 \cdot (p_3^2 + 2p_1p_2) + 1 \cdot (2p_1p_3 + 2p_2p_3) + 2 \cdot (p_1^2 + p_2^2)$$

$$\mathbb{E}[Y] = \sum_{x \in \Omega_X} |x| p_X(x) = |-2|(p_2^2) + |-1|(2p_2p_3) + 0(p_3^2 + 2p_1p_2) + |1|(2p_1p_3) + |2|(p_1^2)$$

They are equal.

(c) Let's give some concrete values: let $p_1 = p_2 = \frac{1}{2}$, and $p_3 = 0$ (so the frog just moves left and right with equal probability, and can't stay still). Demonstrate $\mathbb{E}[\,|\,X|] \neq |\mathbb{E}[X]\,|$ (Recall $Y = |X|$, so $\mathbb{E}[\,|\,X|] = \mathbb{E}[Y]$).

Solution:

$\mathbb{E}[X] = 2(p_1 - p_2) = 0$, so $|\mathbb{E}[X]\,| = 0$ as well. On the other hand, $\mathbb{E}[\,|\,X|] = \mathbb{E}[Y] = 2(\frac{1}{2}^2 + \frac{1}{2}^2) = \frac{1}{2}$. They are not equal.

4. Hungry Washing Machine

You have 10 pairs of socks (so 20 socks in total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks chosen at random. Every subset of 4 socks is equally probable to be the subset that gets eaten. Let X be the number of complete pairs of socks that you have left.

(a) What is the codomain of X, Ω_X (the set of possible values it can take on)? What is the probability mass function of X?

Solution:

$$\Omega_X = \{6, 7, 8\}$$

$$p_X(k) = \begin{cases} \frac{\binom{10}{4}2^4}{\binom{20}{4}} & k = 6 \\ \frac{10\binom{9}{2}2^2}{\binom{20}{4}} & k = 7 \\ \frac{\binom{10}{2}}{\binom{20}{4}} & k = 8 \end{cases}$$

(b) Find $\mathbb{E}[X]$ from the definition of expectation.

Solution:

$$\mathbb{E}[X] = 6 \cdot \frac{\binom{10}{4}2^4}{\binom{20}{4}} + 7 \cdot \frac{10\binom{9}{2}2^2}{\binom{20}{4}} + 8 \cdot \frac{\binom{10}{2}}{\binom{20}{4}} = \frac{120}{19}$$

(c) Find $\mathbb{E}[X]$ using linearity of expectation.

Solution:

For $i \in [10]$, let X_i be 1 if pair i survived, and 0 otherwise. Then, $X = \sum_{i=1}^{10} X_i$. But $\mathbb{E}[X_i] = 1 \cdot \Pr(X_i = 1) + 0 \cdot \Pr(X_i = 0) = \Pr(X_i = 1) = \frac{\binom{18}{4}}{\binom{20}{4}}$. Hence,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{10} X_i\right] = \sum_{i=1}^{10} \mathbb{E}[X_i] = \sum_{i=1}^{10} \frac{\binom{18}{4}}{\binom{20}{4}} = 10\frac{\binom{18}{4}}{\binom{20}{4}} = \frac{120}{19}$$

(d) Which way was easier? Doing both (a) and (b), or just (c)?

Solution:

Part (c).