CSE 312: Foundations of Computing II

QuickCheck: Random Variables, Linearity of Expectation Solutions (due Thursday, April 19)

0. Double the Die, Square the Sum!

Consider the following program:

```
def experiment():
    die1 = RollDie(3)
    die2 = RollDie(2)
    result = (die1 + die2)^2
    return result
```

We want to analyze the return value of this program by modeling it with random variables. Let D_1, D_2, X each represent the value of the first die, the value of the second die, and the returned value.

(a) Find the codomain of X and $p_X(k)$, the probability mass function for X.

Solution:

The codomain of X is $\{2^2, 3^2, 4^2, 5^2\}$.

$$p_X(k) = \begin{cases} \frac{1}{6} & k = 2^2 \\ \frac{2}{6} & k = 3^2 \\ \frac{2}{6} & k = 4^2 \\ \frac{1}{6} & k = 5^2 \end{cases}$$

(b) Find $\mathbb{E}[X]$ by definition of expectation.

Solution:

$$\mathbb{E}[X] = \frac{1}{6} \cdot 2^2 + \frac{2}{6} \cdot 3^2 + \frac{2}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 = \frac{79}{6}$$

(c) Sharpnel insisted on using linearity of expectation to find $\mathbb{E}[X]$. He wrote the following:

$$\mathbb{E}[D_1] = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 3 = 2 \tag{1}$$

$$\mathbb{E}[D_2] = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 2 = 1.5 \tag{2}$$

$$\mathbb{E}[X] = \mathbb{E}\left[(D_1 + D_2)^2\right] \tag{3}$$

$$= \mathbb{E}\left[D_1^2 + 2D_1D_2 + D_2^2\right] \tag{4}$$

$$= (\mathbb{E}[D_1])^2 + 2\mathbb{E}[D_1]\,\mathbb{E}[D_2] + (\mathbb{E}[D_2])^2 \tag{5}$$

$$= 2^2 + 2 \times 2 \times 1.5 + 1.5^2 \tag{6}$$

If he is correct, say so. Else find the first line where he made an error and explain your reasoning briefly.

Solution:

Line 5 is incorrect because $\mathbb{E}[X \cdot Y] \neq \mathbb{E}[X] \cdot \mathbb{E}[Y]$ in general.