CSE 312: Foundations of Computing II

Section 3: Conditional Probability Solutions

0. Allergy Season

In a certain population, everyone is equally susceptible to colds. The number of colds suffered by each person during each winter season ranges from 0 to 4. A new cold prevention drug is introduced that, for people for whom the drug is effective, changes the probabilities of getting a certain number of colds. Unfortunately, the effects of the drug last only the duration of one winter season, and the drug is only effective in 20% of people, independently. The second column of the table shows the probability of getting a certain number of colds given that the drug was ineffective or not taken, and the third column shows the probability of getting a certain number of colds given that the drug was effective.

number of colds	no drug or ineffective	drug effective
0	0.2	0.4
1	0.2	0.3
2	0.2	0.2
3	0.2	0.1
4	0.2	0.0

(a) Sneezy decides to take the drug. Given that he gets 1 cold that winter, what is the probability that the drug is effective for Sneezy?

Solution:

Let E be the event that the drug is effective for Sneezy, and C_i be the event that he gets i colds the first winter. By Bayes' Theorem,

$$\Pr(E \mid C_1) = \frac{\Pr(C_1 \mid E)\Pr(E)}{\Pr(C_1 \mid E)\Pr(E) + \Pr(C_1 \mid \overline{E})\Pr(\overline{E})} = \frac{0.3 \times 0.2}{0.3 \times 0.2 + 0.2 \times 0.8} = \frac{3}{11}$$

(b) The next year he takes the drug again. Given that he gets 2 colds in this winter, what is the updated probability that the drug is effective for Sneezy? (You need to incorporate the information from the previous part, that Sneezy had one cold last winter).

Solution:

Let the reduced sample space for part (b) be C_1 from part (a). Hence, $\Pr(E) = \frac{3}{11}$ instead of 0.2. Let D_i be the event that he gets i colds the second winter. By Bayes' Theorem,

$$\Pr(E \mid D_2) = \frac{\Pr(D_2 \mid E)\Pr(E)}{\Pr(D_2 \mid E)\Pr(E) + \Pr(D_2 \mid \overline{E})\Pr(\overline{E})} = \frac{0.2 \times \frac{3}{11}}{0.2 \times \frac{3}{11} + 0.2 \times \frac{8}{11}} = \frac{3}{11}$$

(c) Why is the answer to (b) the same as the answer to (a)?

Solution:

The probability of two colds whether or not the drug was effective is the same. Hence knowing that Sneezy got two colds does not change the probability of the drug's effectiveness.

1. Balls in Urns

Suppose we have three urns with the following number of red, white, and blue balls in them:

Urn	Red	White	Blue
A	6	5	2
B	4	3	6
C	5	6	7

Suppose we choose an urn by the following rules, after flipping a fair coin three times independently:

(a) If all flips are the same, pick from Urn A

(b) If there is exactly one head, pick from Urn B

(c) Else, pick from Urn C

After choosing an urn, we draw 5 balls without replacement, and let R be the event that exactly three of them are red. Let A, B, C be the events we chose urn A,B,C respectively. What is the probability we chose urn C, given that we drew exactly three of the five balls being red? We'll solve this in three steps.

(a) First, find $\text{Pr}(A), \text{Pr}(B), \text{Pr}(C)$.

Solution:

$$\text{Pr}(A) = \frac{2}{8}, \text{Pr}(B) = \frac{3}{8}, \text{Pr}(C) = \frac{3}{8}$$

(b) Now find $\text{Pr}(R)$, and do not simplify.

Solution:

$$\text{Pr}(R) = \text{Pr}(R \mid A)\,\text{Pr}(A) + \text{Pr}(R \mid B)\,\text{Pr}(B) + \text{Pr}(R \mid C)\,\text{Pr}(C) = \frac{\binom{6}{3}\binom{7}{2}}{\binom{13}{5}} \cdot \frac{2}{8} + \frac{\binom{4}{3}\binom{9}{2}}{\binom{13}{5}} \cdot \frac{3}{8} + \frac{\binom{5}{3}\binom{13}{2}}{\binom{18}{5}} \cdot \frac{3}{8}$$

(c) Finally, compute $\text{Pr}(C \mid R)$, and do not simplify.

Solution:

We have by Bayes Theorem,

$$\text{Pr}(C \mid R) = \frac{\text{Pr}(R \mid C)\,\text{Pr}(C)}{\text{Pr}(R)} = \frac{\frac{\binom{5}{3}\binom{13}{2}}{\binom{18}{5}} \cdot \frac{3}{8}}{\frac{\binom{6}{3}\binom{7}{2}}{\binom{13}{5}} \cdot \frac{2}{8} + \frac{\binom{4}{3}\binom{9}{2}}{\binom{13}{5}} \cdot \frac{3}{8} + \frac{\binom{5}{3}\binom{13}{2}}{\binom{18}{5}} \cdot \frac{3}{8}}$$

2. Infinite Lottery

Suppose we randomly generate a number from the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$. Let A_k be the event we generate the number k, and suppose $\text{Pr}(A_k) = (\frac{1}{2})^k$. Once we generate a number k, that is the maximum we can win. That is, after generating a value k, we can win any number in $[k] = \{1, ..., k\}$ dollars. Suppose the probability that we win $\$j$ for $j \in [k]$ is "uniform", that is, each has probability $\frac{1}{k}$. Let B be the event we win exactly $\$1$. Given that we win exactly one dollar, what is the probability that the number generated was also 1? You may use the fact that $\sum_{j=1}^{\infty} \frac{1}{j \cdot a^j} = \ln(\frac{a}{a-1})$for $a > 1$.

Solution:

$$\mathsf{Pr}(A_1 \mid B) = \frac{\mathsf{Pr}(B \mid A_1)\,\mathsf{Pr}(A_1)}{\sum_{j=1}^{\infty} \mathsf{Pr}(B \mid A_j)\,\mathsf{Pr}(A_j)} = \frac{\frac{1}{1}\frac{1}{2^1}}{\sum_{j=1}^{\infty} \frac{1}{j}\frac{1}{2^j}} = \frac{1}{2\ln 2} \approx \boxed{0.7213}$$