CSE 312: Foundations of Computing II

Section 2: More Combinatorics, Intro Probability Solutions

0. Trick or Treat

Suppose on Halloween, someone is too lazy to keep answering the door, and leaves a jar of exactly N total candies. You count that there are exactly K of them which are kit kats (and the rest are not). The sign says to please take exactly n candies. Each item is equally likely to be drawn. Let X be the number of kit kats we draw (out of n). What is $\Pr(X = k)$, that is, the probability we draw exactly k kit kats?

Solution:

$$\boxed{\Pr(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}}$$

We choose k out of the K kit kats, and $n - k$ out of the $N - K$ other candies. The denominator is the total number of ways to choose n candies out of N total.

1. Staff Photo

Suppose we have 13 chairs (in a row) with 8 TA's, and Professors Blank, Karlin, Ruzzo, Rao, and Tompa to be seated. Suppose all seatings are equally likely. What is the probability that every professor has a TA to his/her immediate left and right?

Solution:

Imagine we permute all 8 TA's first – there are $8!$ ways to do this. Then, there are 7 spots between them, in which we choose 5 for the Professors to sit – order matters since each Professor is distinct so we multiply by $5!$. So the total ways is $\binom{7}{5} \cdot 5! \cdot 8!$.

The total number of ways to seat all 13 of us is simply $13!$.

The probability is then $\boxed{\dfrac{\binom{7}{5} \cdot 5! \cdot 8!}{13!}}$

2. Ingredients

(a) Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent.

Solution:

We use inclusion-exclusion. Let Ω be the set of all anagrams (permutations) of "INGREDIENT", and A_I be the set of all anagrams with two consecutive I's. Define A_E and A_N similarly. $A_I \cup A_E \cup A_N$ clearly are the set of anagrams we don't want. So we use complementing to count the size of $\Omega \setminus (A_I \cup A_E \cup A_N)$. By inclusion exclusion, $|A_I \cup A_E \cup A_N|$ =singles-doubles+triples, and by complementing, $|\Omega \setminus (A_I \cup A_E \cup A_N)| = |\Omega| - |A_I \cup A_E \cup A_N|$.

First, $|\Omega| = \dfrac{10!}{2!2!2!}$ because there are 2 of each of I,E,N's (multinomial coefficient). Clearly, the size of A_I is the same as A_E and A_N. So $|A_I| = \dfrac{9!}{2!2!}$ because we treat the two adjacent I's as one entity. We also need $|A_I \cap A_E| = \dfrac{8!}{2!}$ because we treat the two adjacent I's as one entity and the two adjacent E's as one entity (same for all doubles). Finally, $|A_I \cap A_E \cap A_N| = 7!$ since we treat each pair of adjacent I's, E's, and N's as one entity.

Putting this together gives $\boxed{\frac{10!}{2!2!2!} - \left(\binom{3}{1} \cdot \frac{9!}{2!2!} - \binom{3}{2} \cdot \frac{8!}{2!} + \binom{3}{3} \cdot 7!\right)}$

(b) Repeat the question for the letters "AAAAABBB".

Solution:

For the second question, note that no A's and no B's can be adjacent. So let us put the B's down first: $\_B\_B\_B\_$

By the pigeonhole principle, two A's must go in the same slot, but then they would be adjacent, so there are $\boxed{\text{no ways}}$.

3. Divisibility

Consider the set $T = \{1, 2, ..., 36050\}$, and suppose we choose a subset S of size 3605, each equally likely. What is the probability that there are two (distinct) numbers in S whose difference is divisible by 99?

Solution:

This probability is 1 by the pigeonhole principle. Consider each of the elements of S mod 99 (there are 99 possible remainders $0, ..., 98$). By the pigeonhole principle, since $3605 > 99$, there are at least two with the same remainder. Take those two numbers, and their difference is divisible by 99.