CSE 312: Foundations of Computing II

Section 2: More Combinatorics, Intro Probability (Extra) Solutions

0. Fleas on Squares

25 fleas sit on a 5×5 checkerboard, one per square. At the stroke of noon, all jump across an edge (not a corner) of their square to an adjacent square. At least two must end up in the same square. Why?

Solution:

There are two colors on a checkerboard; so 13 are of one color, and 12 are of another.The 13 fleas must jump to the opposite color of which there are only 12 positions, so at least two fleas must land on the same square by the pigeonhole principle.

1. PigONEholes

Let $k \geq 2$ be some integer. Show that there exists a positive integer n consisting of only digits $0, 1$ and no larger than 10^{k+2} such that $k|n$. (Hint: Consider the sequence of length $k+1$ of $1, 11, 111, 1111, ...$).

Solution:

Consider the sequence of numbers of length $k+1$: $1, 11, 111, ...,$, such that the j^{th} element is the number consisting of exactly j 1's. Take these numbers mod k. Since there are $k+1$ numbers and k possible remainders, two have the same remainder. Call the larger one b and the smaller a. Their difference $n = b - a$ must be divisible by k, and consist of only 1's and 0's.

2. Divide Me

How many numbers in $[360]$ are divisible by:

(a) 4, 6, and 9?

Solution:

This is just the multiplies of $\text{lcm}(4, 6, 9) = 36$. There are $\boxed{\frac{360}{36} = 10}$ multiples.

(b) 4, 6, or 9?

Solution:

We must use inclusion-exclusion. $\boxed{\frac{360}{4} + \frac{360}{6} + \frac{360}{9} - \frac{360}{\text{lcm}(4,6)} - \frac{360}{\text{lcm}(4,9)} - \frac{360}{\text{lcm}(6,9)} + \frac{360}{\text{lcm}(4,6,9)}}$

(c) Neither 4, 6, nor 9?

Solution:

This is just the complement of the previous part, so it is 360 minus the answer to (b).

3. Keep Drawing Cards...

How many cards must you draw from a standard 52-card deck (4 suits and 13 cards of each suit) until you are guaranteed to have:

(a) A single pair? (e.g., AA, 99, JJ)

Solution:

The worst that could happen is to draw 13 different cards, but the next is guaranteed to form a pair. So the answer is $\boxed{14}$.

(b) Two (different) pairs? (e.g., AAKK, 9933, 44QQ)

Solution:

The worst that could happen is to draw 13 different cards, but the next is guaranteed to form a pair. But then we could draw the other two of that pair as well to get 16 still without two pairs. So the answer is $\boxed{17}$.

(c) A full house (a triple and a pair)? (e.g., AAAKK, 99922, 555JJ)

Solution:

The worst that could happen is to draw all pairs (26 cards). Then the next is guaranteed to cause a triple. So the answer is $\boxed{27}$.

(d) A straight (5 in a row, with the lowest being A,2,3,4,5 and the highest being 10,J,Q,K,A)?

Solution:

The worst that could happen is to draw all the $A-4$, $6-9$, and $J-K$. After drawing these $11 \cdot 4$ cards, we could still fail to have a straight. Finally, getting a 5 or 10 would give us a straight. So the answer is $\boxed{45}$.

(e) A flush (5 cards of the same suit)? (e.g., 5 hearts, 5 diamonds)

Solution:

The worst that could happen is to draw 4 of each suit (16 cards), and still not have a flush. So the answer is $\boxed{17}$.

(f) A straight flush (a straight but all cards of the same suit)?

Solution:

Same as getting a straight: $\boxed{45}$.

4. Acing the Exams

In a town of 351 students (the number of students, not ones taking CSE 351), every student aces the midterm, final, or both. If 331 of the students ace the midterm and 45 ace the final, how many people who aced the midterm did not ace the final as well?

Solution:

By inclusion-exclusion, the number of people who aced both the midterm and the final is $331 + 45 - 351 = 25$. For one of the 331 students who aced the midterm; either they aced the final or they didn't, so $331 - 25 = 306$ did not ace the final.

5. Friends

Show that in a group of n people (who may be friends with any number of other people), two must have the same number of friends.

Solution:

There are two cases:

Case 1: Someone has no friends (call that person A)

In this case, for the other $n-1$ people, the possible number of friends if 0 to $n-2$ since they can't be friends with A. If one of them has no friends, we are done, since they have the same number of friends as A. Else they all have 1 to $n-2$ friends, and by the pigeonhole principle, two people have the same number of friends.

Case 2: Everyone has at least one friend Then the number of possible friends is 1 to $n-1$. By the pigeonhole principle, two people have the same number of friends.

6. Senate Committee Assignments

There are 51 senators in a senate. The senate needs to be divided into m committees such that each senator is on exactly one committee. Each senator hates exactly three other senators. (If senator A hates senator B, then senator B does 'not' necessarily hate senator A.) Find the smallest m such that it is always possible to arrange the committees so that no senator hates another senator on his or her committee.

Solution:

We'll prove this statement by induction: for any number of senators n who hate exactly three people, $m = 7$ committees is enough. (and hence, in particular for $n = 51$ senators.)

Base Case: For $n \leq 7$, this is trivial - assign everyone to their own committee.

Induction Hypothesis: Now suppose it is true for any group of n senators ; we will show it is true for $n+1$ senators.

Induction Step: First observe that there are exactly $3n$ pairs (A, B), where A hates B (given). We claim that there exists at least one senator who is hated by at most three people. Why? If not, all senators are hated by > 3 people, a contradiction to our first observation. (This is the Pigeonhole Principle).

Choose one such senator S and separate that senator. For the other n senators, the induction hypothesis says we can arrange them in 7 committees. But senator S hates exactly three people, and is hated by at most three people, so can only be excluded from at most six committees. Place S in one of the remaining committees, and this concludes our induction step.

Minimality: Suppose $m \geq 7$. To show 7 is the least such number, if we only had 6 committees, imagine three senators hating some senator S, and S hating a different set of three senators. Then, S cannot be placed in any committee.

7. Spades and Hearts

Given 3 different spades and 3 different hearts, shuffle them. Compute Pr(E), where E is the event that the suits of the shuffled cards are in alternating order. What is your sample space?

Solution:

The sample space is all reorderings possible: there are $6!$ such. Now order the spades and hearts independently, so there are $3!^2$ ways to do so. Finally choose whether you want hearts or spades first. So Pr$(E) = \frac{2 \cdot 3!^2}{6!}$.

8. Congressional Tea Party

Twenty politicians are having a tea party, 6 Democrats and 14 Republicans.

(a) If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans?

Solution:

The sample space is the number of ways to give tea to people, so there are $\binom{20}{10}$ ways. The event is the ways to give tea to only Republicans, of which there are $\binom{14}{10}$ ways. So the probability is $\frac{\binom{14}{10}}{\binom{20}{10}}$.

(b) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 8 Republicans and 2 Democrats?

Solution:

Similarly to the previous part, $\frac{\binom{14}{8}\binom{6}{2}}{\binom{20}{10}}$.

9. Dinner Party

At a dinner party, the n people present are to be seated uniformly spaced around a circular table. Suppose there is a nametag at each place at the table and suppose that nobody sits down at the correct place. Show that it is possible to rotate the table so that at least two people are sitting in the correct place.

Solution:

For $i = 1, ..., n$, let r_i be the number of rotations clockwise needed for the i^{th} person to be in their spot. Each r_i can be between 1 and $n - 1$ (not 0 since no one is at their nametag, and not n since it is equivalent to 0). Since there are n people and only $n - 1$ possible values for the rotations, at least two must have the same value by the pigeonhole principle. Rotate the table clockwise by that much, and at least two people will be in the correct place.