CSE 312: Foundations of Computing II

Section 1: Combinatorics Solutions

0. Birthday Cake

A chef is preparing desserts for the week, starting on a Sunday. On each day, only one of five desserts (apple pie, cherry pie, strawberry pie, pineapple pie, and cake) may be served. On Thursday there is a birthday, so cake must be served that day. On no two consecutive days can the chef serve the same dessert. How many dessert menus are there for the week?

Solution:

Start from Thursday and work forward and backward in the week: $4 \cdot 4 \cdot 4 \cdot 4 \cdot 1 \cdot 4 \cdot 4 = \boxed{4^6}$

1. Photographs

Suppose that 8 people, including you and a friend, line up for a picture. In how many ways can the photographer organize the line if she wants to have fewer than 2 people between you and your friend?

Solution:

There are two cases; so, we count using the rule of sum:

Case 1: You are next to your friend. Then there are there are 7 different slots. Then, there are 7 sets of positions you and your friend can occupy (positions 1/2, 2/3, ..., 7/8), and for each set of positions, there are 2 ways to arrange you and your friend, So there are $7 \cdot 2$ ways to pick positions for you and your friend.

Case 2: There is exactly 1 person between you and your friend. Then, there are 6 sets of positions you and your friend can occupy (positions 1/3, 2/4, ... , 6/8), and for each set of positions, there are 2 ways to arrange you and your friend. So there are $6 \cdot 2$ ways to pick positions for you and your friend.

Note that in both cases, there are then $6!$ ways to arrange the remaining people, so we multiply both cases by $6!$.

Therefore, the answer is $\boxed{(2 \cdot 7 + 2 \cdot 6) \cdot 6!}$

2. Rearrangements

Permutations of objects, some of which are indistinguishable.

(a) How many permutations are there of the letters in DAWGY?

Solution:

$\boxed{5!}$

(b) How many permutations are there of the letters in DOGGY?

Solution:

Let x be the number of permuations there are of the letters in DOGGY. Notice that if we consider the two Gs as distinct letters, (say, by labelling them G_1 and G_2), then there are $5!$ ways of rearranging the [now] 5 distinct letters of DOG_1G_2Y. However, we can count the same thing by noticing that when we rearrange the letters in DOGGY, there are two possible orderings of Gs for each rearrangement: either the G_1 can come first or the G_2. Therefore, there are also $2x$ ways to rearrange DOG_1G_2Y. Thus, we see

$$\boxed{5! = 2x, x = \frac{5!}{2}}$$

(c) How many permutations are there of the letters in GODOGGY?

Solution:

Like in the previous question, let x be the number of permuations there are of the letters in GODOGGY. Notice that if we consider the three Gs and two Os as distinct letters, (say, by labelling them G_1 and G_2), then there are $5!$ ways of rearranging the [now] 7 distinct letters of $\text{DOG}_1\text{G}_2\text{Y}$. However, we can count the same thing by noticing that when we rearrange the letters in GODOGGY, there are $3!$ possible orderings of Gs for each rearrangement and $2!$ possible reorderings of Os for each rearrangement. Therefore, there are also $3! \cdot 2 \cdot x$ ways to rearrange $\text{G}_1\text{O}_1\text{DO}_2\text{G}_2\text{G}_3\text{Y}$. Thus, we see $\boxed{7! = 3! \cdot 2 \cdot x, x = \frac{7!}{3! \cdot 2!}}$

3. Rabbits!

Rabbits Peter and Pauline have three offspring: Flopsie, Mopsie, and Cotton-tail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

Solution:

If Peter and Pauline go to the same store, there are 4 stores it could be. For each such choice, there are 3 choices of store for each of the 3 offspring, so 3^3 choices for all the offspring. If Peter and Pauline go to different stores, there are $4 \cdot 3 = 12$ pairs of stores they could go to. For each such choice, there are 2 choices of store for each of the 3 offspring, so 2^3 choices for all the offspring. Therefore the answer is $\boxed{4 \cdot 3^3 + 12 \cdot 2^3}$.

4. Seating

How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if . . .

(a) . . . the seats are assigned arbitrarily?

Solution:

$\boxed{10!}$

(b) . . . all couples are to get adjacent seats?

Solution:

$\boxed{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2 = 2^5 \cdot 5!}$: there are $5!$ permutations of the 5 couples, and then 2 permutations within each of the 5 couples.

(c) . . . the seats are assigned arbitrarily, except that one couple insists on not sitting in adjacent seats?

Solution:

There are $9! \cdot 2$ arrangements in which this couple does sit in adjacent seats, since you can treat the couple as a ninth unit added to the other 8 individuals, and then there are 2 permutations of that couple's seats. That means the answer to the question is $\boxed{10! - 9! \cdot 2 = 8 \cdot 9!}$.

Alternatively, we can do casework. Name the two people in the couple A and B. There are two cases: A can sit on one of the ends, or not. If A sits on an end seat, A has 2 choices and B has 8 possible seats. If A doesn't sit on the end, A has 8 choices and B only has 7. So there are a total of $2 \cdot 8 + 8 \cdot 7$ ways A and B can sit. Once they do, the other 8 people can sit in $8!$ ways since there are no other restrictions. Hence the total number of ways is $\boxed{(2 \cdot 8 + 8 \cdot 7)8! = 9 \cdot 8 \cdot 8! = 8 \cdot 9!}$.

5. Bridge

How many bridge hands have a suit distribution of 5, 5, 2, 1? (That is, you are playing with a standard 52-card deck and you have 5 cards of one suit, 5 cards of another suit, 2 of another suit, and 1 of the last suit.)

Solution:

The number of hands here consists of two parts: first, we must count the number of ways to choose 5, 5, 2, and 1 card[s] from 4 suits, each of which has 13 cards. These are all separate choices, so we simply multiply $\binom{13}{5} \cdot \binom{13}{5} \cdot \binom{13}{2} \cdot \binom{13}{1}$. Now, we must assign from which two suits we choose 5 cards, and from which we choose 2, and which we choose only 1. We first choose the 2 suits out of 4 where we choose 5 cards - this is just $\binom{4}{2}$. Finally, there are 2 ways to assign the remaining two suits to 2 different sets of cards - applying the product rule to all this gives the below:

$\boxed{\binom{13}{5}\binom{13}{5}\binom{13}{2}\binom{13}{1} \cdot \binom{4}{2} \cdot 2}$