CSE 312: Foundations of Computing II

Section 1: Combinatorics (Extra) Solutions

0. Weird Card Game

In how many ways can a pack of fifty-two cards be dealt to thirteen players, four to each, so that every player has one card of each suit?

Solution:

Deal one suit at a time. For each suit, there are $13!$ ways to distribute one card to each person. So the answer is $\boxed{13!^4}$.

1. Escape the Professor

There are 6 security professors and 7 theory professors taking part in an escape room. If 4 security professors and 4 theory professors are chosen and paired off, how many pairings are possible?

Solution:

$\boxed{\binom{6}{4}\binom{7}{4}4!}$. First choose 4 of the security professors, then 4 of the theory professors. Then assign each theory professor to a security professor.

2. Names

How many ways are there to choose three initials (upper case letters) such that two are the same or all three are the same?

Solution:

Complementary counting: Count the total 26^3 and subtract the number with all distinct initials $26 \cdot 25 \cdot 24$ to get $\boxed{26^3 - 26 \cdot 25 \cdot 24}$.

3. Extended Family Portrait

A group of n families, each with m members, are to be lined up for a photograph. In how many ways can the nm people be arranged if members of a family must stay together?

Solution:

First order the families, of which there are $n!$ ways. Within each family, there are $m!$ ways to order their members. So there are a total of $\boxed{n!(m!)^n}$ ways.

4. Full Class

There are 40 seats and 40 students in a classroom. Suppose that the front row contains 10 seats, and there are 5 students who must sit in the front row in order to see the board clearly. How many seating arrangements are possible with this restriction?

Solution:

$\boxed{\binom{10}{5} \cdot 5! \cdot 35!}$ Seat the students who must sit in the front row first. There are $\binom{10}{5} \cdot 5!$ ways to assign seats to those students, since we choose any 5 of the 10 seats, and then assign them. Then there are 35 students and 35 seats left, so there are $35!$ ways to assign seats to the other students

5. HBCDEFGA

How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

Solution:

First, we assign A not to the beginning. There are two cases:

Case 1: A is at the end

In this case, the remaining 7 letters can go anywhere, so there are $7!$ arrangements.

Case 2: A is neither at the end nor the beginning

There are 6 choices for A if it is not at either end. Then, there are 6 remaining valid spots for H (cannot be at the end). Finally the other 6 letters can be arranged in any way. So we have $6^2 \cdot 6!$ ways.

Hence the total number of ways is $\boxed{7! + 6^2 \cdot 6!}$.

6. Graduation Planning

Suppose you have five quarters left and you want to take exactly two classes per quarter. You want to take CSE1, CSE2,..., CSE10, but CSE1 and CSE2 must be taken before CSE3, which must be taken before CSE4. CSE1 and CSE2 can be taken in any order, or together. The other classes can be taken any quarter. How many different schedules can be formed (assume the two classes in a quarter are unordered)?

Solution:

We must split cases:

Case 1: CSE1 and CSE2 are taken in the same quarter

In this case, we have the sequence CSE1/2-CSE3-CSE4. There are $\binom{5}{3}$ ways to choose which 3 of the 5 quarters this sequence is taken in. Then there are 6 classes which can be taken with CSE3 and then 5 which can be taken with CSE4. Finally, there are two quarters remaining and four classes; there are $\binom{4}{2}$ ways to assign two classes to one of them, and the other has no choice. So there are $\binom{5}{3} \cdot 6 \cdot 5 \cdot \binom{4}{2}$ schedules.

Case 2: CSE1 and CSE2 are taken in different quarters

In this case, CSE1-4 are taken in a different quarter each. So we have the sequence CSE1-CSE2-CSE3-CSE4 or CSE2-CSE1-CSE3-CSE4. There are $\binom{5}{4}$ ways to choose which four quarters this sequence falls in. Then there are $6 \cdot 5 \cdot 4 \cdot 3$ ways to assign classes to pair with this sequence. The last two classes have no choice. So the number of schedules here is $2 \cdot \binom{5}{4} \cdot 6 \cdot 5 \cdot 4 \cdot 3$.

Adding these disjoint cases gives $\boxed{\binom{5}{3} \cdot 6 \cdot 5 \cdot \binom{4}{2} + 2 \cdot \binom{5}{4} \cdot 6 \cdot 5 \cdot 4 \cdot 3}$. different schedules.

7. Paired Finals

Suppose you are to take the CSE 312 final in pairs. There are 100 students in the class and 8 TAs, so 8 lucky students will get to pair up with a TA. Each TA must take the exam with some student, but two TAs cannot take the exam together. How many ways can they pair up?

Solution:

First we choose the 8 lucky students, and then pair them with a TA. There are $\binom{100}{8}$ ways to choose the students, and $8!$ ways to pair them with the TAs. There are 92 students left. The first one has 91 choices. Then there are 90 students left. The next one has 89 choices. And so on. So the total number of ways is $\boxed{\binom{100}{8} \cdot 8! \cdot 91 \cdot 89 \cdot ... \cdot 3 \cdot 1}$.