CSE 312

Foundations of Computing II

Random Variables

x

q

r

z

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variable = 17
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$[n] = \{1, 2, \ldots, n\}$

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Looking at the examples above:

- $\texttt{r\_var\_1} : [2] \to [2]$

$\Omega = [2] \times [3]$ $\quad \Omega \longrightarrow [5]$

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We often want to talk about the **probability mass function** of a random variable:

$$p_{\texttt{r\_var\_1}}(x) = \Pr(\texttt{r\_var\_1} = x) = \frac{1}{2} \text{ (for } x \in [2])$$

$$p_{\text{r var 2}}(x) = \frac{1}{3} \text{ for } x = 1, 2, 3$$

$$p_{\text{r var 3}}(x) = \begin{cases} \frac{1}{6} & \text{for } x = 2, 5 \\ \frac{2}{6} & \text{for } x = 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

	1	2
1	2	3
2	3	4
3	4	5

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So, to recap, a random variable is a variable in a program that depends on a random process.

Definition (Random Variable)

A Little More Formal Now. . .

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A **random variable** is a variable in a randomized piece of code **or**
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The **expected value** of a random variable $X : \Omega \to \mathbb{N}$ with p.m.f. $p_X(\cdot)$ is the **weighted** average value it takes on.

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$$\mathbb{E}[X] = \sum_{s \in \Omega} X(s) \Pr(s) = \sum_{n=0}^{\infty} n p_X(n)$$

$$\frac{1}{n}$$

$$\frac{x_1}{n} + \frac{x_2}{n} + \cdots = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

- No student will ever get below 80.
- 20% of the students got a 100.
- 30% of the students got an 80.
- The remaining 50% of the students got scores evenly distributed between $81, 82, \ldots, 99$.

Let X be the r.v. for a student's score in CSE 001.

$$\mathbb{E}[X] = \sum n \cdot p_X(n)$$

$$p_X(n) = \{$$

Average grade in course = $100 \cdot \frac{20}{100} \cdot n + 80 \cdot \frac{30}{100} \cdot n$

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$$p_X(i) = \Pr(X = i) = \begin{cases} \\ \\ \end{cases}$$

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$$p_X(i) = \Pr(X = i) = \begin{cases} 0.3 & \text{if } i = 80 \\ 0.5\frac{1}{19} & \text{if } 81 \le i \le 99 \\ 0.2 & \text{if } i = 100 \\ 0 & \text{otherwise} \end{cases}$$

To get the expected value, we just use the formula:

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To get the expected value, we just use the formula:

$$\mathbb{E}[X] = \sum_{i=0}^{100} i p_X(i) = 100 p_X(100) + 80 p_X(80) + \sum_{i=81}^{99} i p_X(i)$$

$$= (100)(0.2) + (80)(0.3) + \sum_{i=81}^{99} i \frac{0.5}{19}$$

Bernoulli Random Variables


```
if FlipCoin(p) == HEADS:
    X = 1
else:
    X = 0
```

$$\mathbb{E}[X] = 1 \cdot p + 0 \cdot (1-p) + \sum_{n=2}^{\infty} n \cdot P_X(n)$$

$$\sum_{x \in \mathbb{N}} i P_X(i)$$

Bernoulli Random Variables


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if FlipCoin(p) == HEADS:
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We consider HEADS to be a “success” and TAILS to be a “failure”. Notice that the p.m.f. of X is

$$p_X(x) = \Pr(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

If we have a r.v. distributed like X, we say $X \sim \text{Bernoulli}(p)$.

What is $\mathbb{E}[X]$?

$$\mathbb{E}[X] =$$

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What is $\mathbb{E}[X]$?

$$\mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p$$

```
for i = 1 to n:
    if FlipCoin(p) == HEADS:
        X_i = 1
    else:
        X_i = 0
```

$$Y = \sum_{i=1}^{n} X_i$$

$$P_Y(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{s \in \Omega} Y(s) \Pr(s) \quad p^k (1-p)^{n-k}$$

$$E[Y] = \sum_{k \in \mathbb{N}} k P_Y(k) =$$

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for i = 1 to n:
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Notice the following facts:

- $X_i \sim \text{Bernoulli}(p)$
- All the coin flips are unrelated; so, the X_i's are independent.

Consider the r.v. $Y = \sum_{i=1}^{n} X_i$.

When Y is the sum of n Bernoulli distributed r.v.'s, we say $Y \sim \text{Binomial}(n, p)$.

What is $p_Y(\cdot)$?

$$p_Y(k) = \Pr(Y = k) =$$

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Binomial Random Variables

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$\mathbb{E}[\Sigma] = \sum \mathbb{E}$

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What is $\mathbb{E}[Y]$?

$$\mathbb{E}[Y] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np \text{ (WTF?)}$$

```
Z = 0
while FlipCoin(p) != HEADS:
    Z += 1
Z += 1
```

When Z is the number of coin flips up to and including the first HEADS, we say $Z \sim \text{Geometric}(p)$.

What is $p_Y(\cdot)$?

$$p_Y(k) = \Pr(Y = k) =$$

$$p_Z(k) = (1-p)^{k-1} p$$

$$\sum k (1-p)^{k-1} p$$

Geometric Random Variables

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What is $\mathbb{E}[Y]$?

$$\mathbb{E}[Y] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p = p \sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{p}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}$$

Insight: This looks a lot like $\sum_{k=1}^{\infty} kq^{k-1}$ which looks a lot like $\sum_{k=0}^{\infty} q^k = \frac{1}{1-x}$.
Take the derivative of both sides:

$$\sum_{k=1}^{\infty} kq^{k-1} = \frac{d}{dx}\left(\sum_{k=0}^{\infty} q^k\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}$$

Sum of Two Dice Rolls

Consider the experiment:

```
A = RollDie(3)
B = RollDie(3)
C = A + B
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What is $p_C(\cdot)$?

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$$\mathbb{E}[C] =$$

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What is $\mathbb{E}[C]$?

$$\mathbb{E}[C] = 2 \times \frac{1}{9} + 3 \times \frac{2}{9} + 4 \times \frac{3}{9} + 5 \times \frac{2}{9} + 6 \times \frac{1}{9} = 4$$

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What is $\mathbb{E}\left[C^2\right]$?

$$\mathbb{E}\left[C^2\right] = \sum_{c=2}^{6} c^2 p_C(c) = 2^2 \times \frac{1}{9} + 3^2 \times \frac{2}{9} + 4^2 \times \frac{3}{9} + 5^2 \times \frac{2}{9} + 6^2 \times \frac{1}{9} = \frac{52}{3} \approx 17.33$$