

# CSE 312

## Foundations of Computing II

# Random Variables

$x$

$q$

$r$

$z$

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1 variable = 17
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Formally, we can view a random variable  $X$  as a function from  $\Omega$  to  $S$ .

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$$[n] = \{1, 2, \dots, n\}$$

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Looking at the examples above:

$$\blacksquare r\_var\_1 : [2] \rightarrow [2]$$

$$\Omega = [2] \times [3] \quad \Omega \rightarrow [5]$$



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We often want to talk about the **probability mass function** of a random variable:

$$p_{r\_var\_1}(x) = \Pr(r\_var\_1 = x) = \frac{1}{2} \text{ (for } x \in [2])$$

$$p_{r\_var\_2}(x) = \frac{1}{3} \text{ for } x=1,2,3$$

$$p_{r\_var\_3}(x) = \begin{cases} \frac{1}{6} & \text{for } x=2,5 \\ \frac{2}{6} & \text{for } x=3,4 \\ 0 & \text{otherwise} \end{cases}$$

	1	2
1	2	3
2	3	4
3	4	5

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$$p_{\text{r\_var\_3}}(x) = \Pr(\text{r\_var\_3} = x) = \left\{ \begin{array}{l} \frac{1}{6} \quad \text{for } x \in [2, 3] \\ \frac{1}{3} \quad \text{for } x \in [3, 4] \\ \frac{1}{6} \quad \text{for } x \in [4, 5] \end{array} \right.$$

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So, to recap, a random variable is a variable in a program that depends on a random process.



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$$\mathbb{E}[X] = \sum_{s \in \Omega} X(s) \Pr(s) = \sum_{n=0}^{\infty} n p_X(n)$$

$$\frac{x_1}{n} + \frac{x_2}{n} + \dots = \frac{x_1 + x_2 + \dots + x_n}{n}$$

- No student will ever get below 80.
- 20% of the students got a 100.
- 30% of the students got an 80.
- The remaining 50% of the students got scores evenly distributed between 81, 82, ..., 99.

Let  $X$  be the r.v. for a student's score in CSE 001.

$$E[X] = \sum n \cdot p_x(n) \quad \rightarrow \quad p_x(n) = \begin{cases} \frac{1}{10} & n = 100 \\ \frac{3}{100} & n = 80 \\ \frac{1}{100} & n = 81, 82, \dots, 99 \end{cases}$$

$$\text{Average grade in course} = \frac{100 \cdot \frac{20}{100} \cdot n + 80 \cdot \frac{30}{100} \cdot n}{n} = 86$$

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Let  $X$  be the r.v. for a student's score in CSE 001. Then, the p.m.f. of  $X$  is:

$$p_X(i) = \Pr(X = i) = \left\{ \begin{array}{ll} 0.2 & \text{if } i = 100 \\ 0.3 & \text{if } i = 80 \\ 0.01 & \text{if } i = 81, 82, \dots, 99 \\ 0 & \text{otherwise} \end{array} \right.$$

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To get the expected value, we just use the formula:

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$$\begin{aligned} \mathbb{E}[X] &= \sum_{i=0}^{100} i p_X(i) = 100 p_X(100) + 80 p_X(80) + \sum_{i=81}^{99} i p_X(i) \\ &= (100)(0.2) + (80)(0.3) + \sum_{i=81}^{99} i \frac{0.5}{19} \end{aligned}$$



```

1 if FlipCoin(p) == HEADS:
2     X = 1
3 else:
4     X = 0

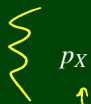
```

$$E[X] = 1 \cdot p + \overset{1}{0} \cdot (1-p) + \sum_{n=2}^{\infty} n \cdot \overset{0}{P_X(n)}$$

$$\sum_{x \in \mathcal{X}} x P_X(x)$$

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We consider HEADS to be a “success” and TAILS to be a “failure”. Notice that the p.m.f. of  $X$  is


$$p_X(x) = \Pr(X = x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \end{cases}$$

If we have a r.v. distributed like  $X$ , we say  $X \sim \text{Bernoulli}(p)$ .

What is  $\mathbb{E}[X]$ ?

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$$\mathbb{E}[X] = 1 \times p + 0 \times (1 - p) = p$$

```

1 for i = 1 to n:
2   if FlipCoin( $p$ ) == HEADS:
3      $X_i = 1$ 
4   else:
5      $X_i = 0$ 

```

$$Y = \sum_{i=1}^n X_i$$

$$P_Y(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{s \in \mathcal{S}} Y(s) \Pr(s) p^k (1-p)^{n-k}$$

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Notice the following facts:

- $X_i \sim \text{Bernoulli}(p)$
- All the coin flips are unrelated; so, the  $X_i$ 's are independent.

Consider the r.v.  $Y = \sum_{i=1}^n X_i$ .

When  $Y$  is the sum of  $n$  Bernoulli distributed r.v.'s, we say  $Y \sim \text{Binomial}(n, p)$ .

What is  $p_Y(\cdot)$ ?

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$$\mathbb{E}[Z] = \sum \mathbb{E}$$

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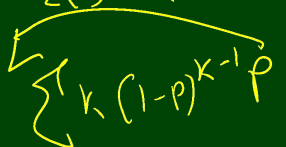
$$\mathbb{E}[Y] = \sum_{k=0}^n k \binom{n}{k} p^k (1-p)^{n-k} = np \text{ (WTF?)}$$

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1 Z = 0
2 while FlipCoin(p) != HEADS:
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4 Z += 1
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When  $Z$  is the number of coin flips up to and including the first HEADS, we say  $Z \sim \text{Geometric}(p)$ .

What is  $p_Z(\cdot)$ ?

$$p_Z(k) = \Pr(Z = k) =$$

$$p_Z(k) = (1-p)^{k-1}p$$

$$\sum_k (1-p)^{k-1}p$$



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What is  $\mathbb{E}[Y]$ ?

$$\mathbb{E}[Y] = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = p \sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{p}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}$$

Insight: This looks a lot like  $\sum_{k=1}^{\infty} kq^{k-1}$  which looks a lot like  $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$ .

Take the derivative of both sides:

$$\sum_{k=1}^{\infty} kq^{k-1} = \frac{d}{dq} \left( \sum_{k=0}^{\infty} q^k \right) = \frac{d}{dq} \left( \frac{1}{1-q} \right) = \frac{1}{(1-q)^2}$$

Consider the experiment:

- 1  $A = \text{RollDie}(3)$
- 2  $B = \text{RollDie}(3)$
- 3  $C = A + B$

What is  $p_C(\cdot)$ ?

$$p_C(x) = \Pr(C = x) = \left\{ \right.$$

Consider the experiment:

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What is  $p_C(\cdot)$ ?

$$p_C(x) = \Pr(C = x) = \begin{cases} \frac{1}{9} & \text{if } x = 2 \\ \frac{2}{9} & \text{if } x = 3 \\ \frac{3}{9} & \text{if } x = 4 \\ \frac{2}{9} & \text{if } x = 5 \\ \frac{1}{9} & \text{if } x = 6 \\ 0 & \text{otherwise} \end{cases}$$

What is  $\mathbb{E}[C]$ ?

$$\mathbb{E}[C] =$$

Consider the experiment:

- 1  $A = \text{RollDie}(3)$
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What is  $p_C(\cdot)$ ?

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What is  $\mathbb{E}[C]$ ?

$$\mathbb{E}[C] = 2 \times \frac{1}{9} + 3 \times \frac{2}{9} + 4 \times \frac{3}{9} + 5 \times \frac{2}{9} + 6 \times \frac{1}{9} = 4$$

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What is  $\mathbb{E}[C^2]$ ?

$$\mathbb{E}[C^2] = \sum_{c=2}^6 c^2 p_C(c) = 2^2 \times \frac{1}{9} + 3^2 \times \frac{2}{9} + 4^2 \times \frac{3}{9} + 5^2 \times \frac{2}{9} + 6^2 \times \frac{1}{9} = \frac{52}{3} \approx 17.33$$