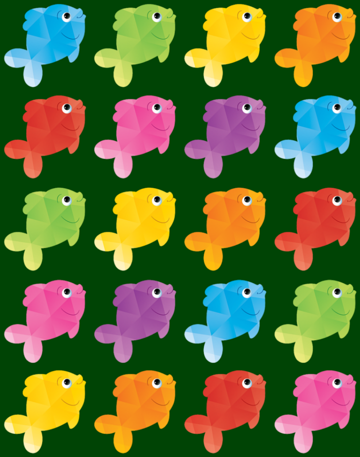
CSE 312

Foundations of Computing II

Poisson Distribution



Uniform Random Variables

```
U = (a − 1) + RollDie(b − a + 1)
```

What is $p_U(\cdot)$?

$$p_U(k) = \Pr(U = k) =$$

What is $\mathbb{E}[U]$?

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What is $\mathrm{Var}(U)$?

$$\mathrm{Var}(U) = \frac{(b-a)(b-a+2)}{12}$$

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U = (a - 1) + RollDie(b - a + 1)
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What is $p_U(\cdot)$?

$$p_U(k) = \Pr(U = k) = \frac{1}{b - a + 1}$$

What is $\mathbb{E}[U]$?

$$\mathbb{E}[U] = \sum_{k=a}^{b} \frac{k}{b-a+1} = \frac{1}{2}(a+b)$$

What is $\mathrm{Var}(U)$?

$$\mathrm{Var}(U) = \frac{(b-a)(b-a+2)}{12}$$

Define a potential distribution with pmf:

$$p_P(k) = \Pr(P = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

Is this a valid distribution?

$$\sum_{k=0}^{\infty} p_P(k) =$$

What is $\mathbb{E}[P]$?

$$\mathbb{E}[P] = \sum_{k=0}^{\infty} k p_P(k) =$$

What is $\mathrm{Var}(P)$?

$$\mathbb{E}\left[P^2\right] = \sum_{k=0}^{\infty} k^2 p_P(k) = \lambda^2 + \lambda$$

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$$\sum_{k=0}^{\infty} p_P(k) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1$$

What is $\mathbb{E}[P]$?

$$\mathbb{E}[P] = \sum_{k=0}^{\infty} k p_P(k) = \sum_{k=1}^{\infty} e^{-\lambda}\frac{k\lambda^k}{k!} = e^{-\lambda}\lambda\sum_{k=1}\frac{\lambda^{k-1}}{(k-1)!} = e^{-\lambda}\lambda e^{\lambda} = \lambda$$

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$$\mathrm{Var}(P) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

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Suppose n potential occurrences of an event happen over a given period of time at an average rate of λ. Let X be the number of actual occurrences of the event.

Note that $X \sim \text{Binomial}\left(n, \frac{\lambda}{n}\right)$.

In this example, only a **fixed** number of events can occur in our time period. Can we model what would happen if an **arbitrary** number of events could occur in a single period of time?

Define Y to be a r.v. with pmf: $p_Y(k) = \lim_{n\to\infty} p_X(k)$. That is, to represent a single time step with an arbitrarily large number of events, we take $n \to \infty$.

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$$
\begin{aligned}
\Pr(Y = k) &= \lim_{n\to\infty} \Pr(X = k) \\
&= \lim_{n\to\infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n!}{(n-k)!} \left(\frac{1}{n^k}\right) \left(1 + \frac{-\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!} \lim_{n\to\infty} \frac{n(n-1)\cdots(n-k+1)}{n^k} \left(1 + \frac{-\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!} \lim_{n\to\infty} \left(1 + \frac{-\lambda}{n}\right)^{n-k} \\
&= \frac{\lambda^k}{k!} \lim_{n\to\infty} \left(1 + \frac{-\lambda}{n}\right)^{n} \\
&= e^{-\lambda} \frac{\lambda^k}{k!}
\end{aligned}
$$

$$p_P(k) = \Pr(P = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

You get email at a rate of $\lambda = 0.2$ messages per hour.

Example

You check your e-mail every hour. What is the probability of finding 0 new messages? What about 1 message?

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- $p_X(0) = e^{-0.2} \approx 0.819$
- $p_X(1) = e^{-0.2}(0.2) \approx 0.164$

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You don't check your e-mail for a whole day. What is the probability of finding 0 new messages?

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Example

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Suppose a server can process k requests per second. Requests arrive at random at an average rate of $\lambda = 1$ per second. What size does k need to be to guarantee a less than 50% chance that we drop a packet?

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