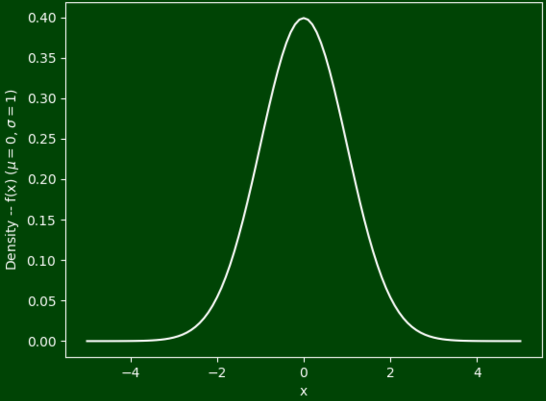
CSE 312

Foundations of Computing II

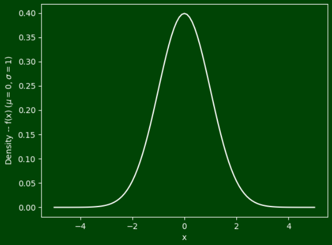
Normal Distribution

PDF

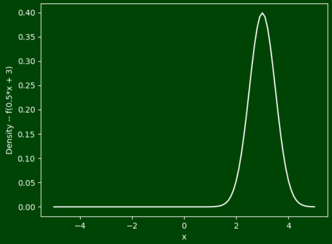
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$



$X \sim \mathcal{N}(0,1)$ PDF



$\frac{X}{2} + 3 \sim \mathcal{N}(3, 0.25)$ PDF



$$\begin{array}{ccc} X & \rightsquigarrow & \frac{X}{2} + 3 \\ \mathcal{N}(0,1) & \rightsquigarrow & \mathcal{N}(3, 0.25) \end{array}$$

$$Y = aX + b$$

- $\mathbb{E}[Y] =$

Linear Transformation of the Normal Distribution

$$Y = aX + b$$

- $\mathbb{E}[Y] = \mathbb{E}[aX + b] = a\mathbb{E}[X] + b = a\mu + b$
- $\mathsf{Var}(Y) = \mathsf{Var}(aX + b) = a^2\mathsf{Var}(x) = a^2\sigma^2$

$$
\begin{aligned}
f_Y(x) = \frac{d}{dx}F_Y(x) = \frac{d}{dx}\mathsf{Pr}(Y \le x) &= \frac{d}{dx}\mathsf{Pr}(aX + b \le x) \\
&= \frac{d}{dx}\mathsf{Pr}\left(X \le \frac{x-b}{a}\right) \\
&= \frac{d}{dx}F_X\left(\frac{x-b}{a}\right) \\
&= \frac{1}{a}f_X\left(\frac{x-b}{a}\right) \\
&= \frac{1}{a}\left(\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-\left(\frac{x-b}{a}-\mu\right)^2}{2\sigma^2}}\right) \\
&= \frac{1}{a\sigma\sqrt{2\pi}}e^{\frac{-\left(\frac{x-b}{a}-\frac{a\mu}{a}\right)^2}{2\sigma^2}} \\
&= \frac{1}{a\sigma\sqrt{2\pi}}e^{\frac{-(x-(a\mu+b))^2}{2a^2\sigma^2}}
\end{aligned}
$$

Standardizing a Normal Distribution

$$X \sim \mathcal{N}(\mu_X, \sigma_X^2) \longrightarrow Z \sim \mathcal{N}(0, 1)$$

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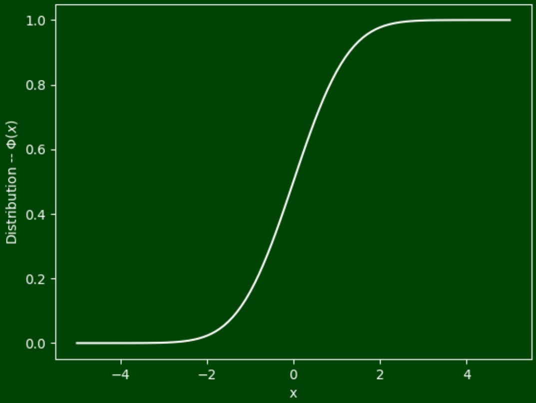
$$X \sim \mathcal{N}(\mu_X, \sigma_X^2) \longrightarrow Z \sim \mathcal{N}(0,1)$$

$$Z = \frac{X - \mu_X}{\sigma_X}$$

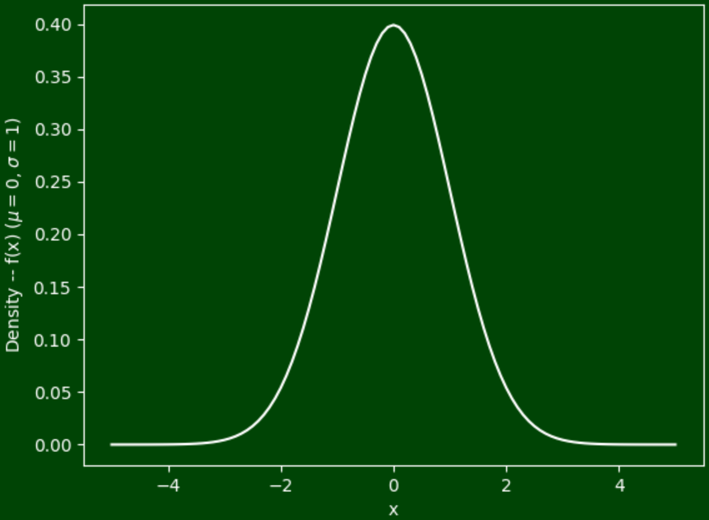
- $\mathbb{E}[Z] = \mathbb{E}\left[\frac{X-\mu_X}{\sigma_X}\right] = \frac{\mathbb{E}[X]-\mu_X}{\sigma_X} = 0$
- $\mathsf{Var}(Z) = \mathsf{Var}\left(\frac{X-\mu_X}{\sigma_X}\right) = \frac{\mathsf{Var}(X)}{\sigma_X^2} = 1$

CDF

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}}\, dt$$



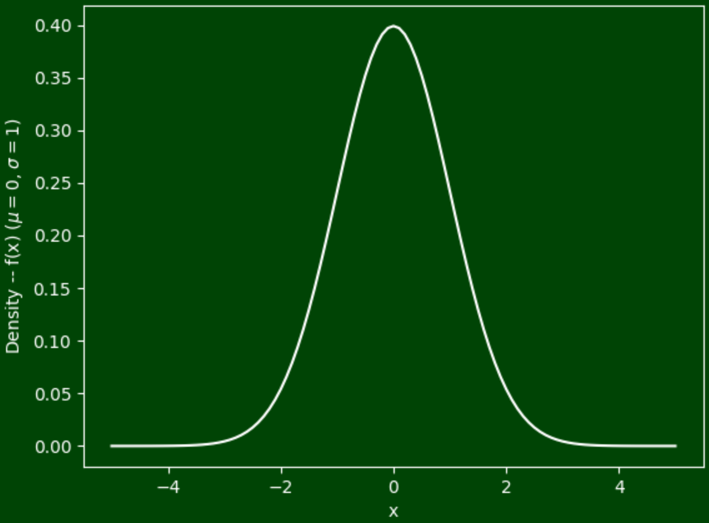
Bounding The Density of a Normal



$$\Pr(\mu - \sigma < Z < \mu + \sigma) = \Phi(1) - \Phi(-1) \approx 68\%$$

$$\Pr(\mu - 2\sigma < Z < \mu + 2\sigma) = \Phi(2) - \Phi(-2) \approx 95\%$$

$$\Pr(\mu - 3\sigma < Z < \mu + 3\sigma) = \Phi(3) - \Phi(-3) \approx 99\%$$


0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
Density -- f(x) ($\mu = 0$, $\sigma = 1$)
−4
−2
0
2
4
x

Suppose X is a non-negative r.v.; then...

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$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=0}^{\alpha-1} k p_X(k) \qquad + \sum_{k=\alpha}^{\infty} k p_X(k) \\
&\geq 0 \qquad\qquad\qquad + \sum_{k=\alpha}^{\infty} k p_X(k) \\
&\geq 0 \qquad\qquad\qquad + \sum_{k=\alpha}^{\infty} \alpha p_X(k) \\
&= \alpha \sum_{k=\alpha}^{\infty} p_X(k) \\
&= \alpha \Pr(X \geq \alpha) \\
\Pr(X \geq \alpha) &\leq \frac{\mathbb{E}[X]}{\alpha}
\end{aligned}$$

Chebyshev's Inequality

Suppose X is a non-negative r.v.; then. . .

Suppose X is a non-negative r.v.; then...

$$\begin{aligned}\Pr(|X - \mathbb{E}[X]| \geq \alpha) &= \Pr\big((X - \mathbb{E}[X])^2 \geq \alpha^2\big) \\ &\leq \frac{\mathbb{E}\big[(X - \mathbb{E}[X])^2\big]}{\alpha^2} \\ &= \frac{\mathrm{Var}(X)}{\alpha^2}\end{aligned}$$