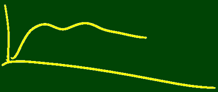
CSE 312

Foundations of Computing II

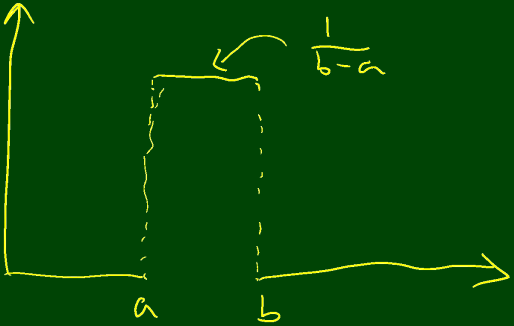
CDF

$$\Pr(X \le k) = \int_{-\infty}^{k} f_X(x)\,dx$$

PDF



More Continuous Distributions



Binomial (n, p)

The number of times an event occurs from n possible occurences, each event with probability p.

Poisson (λ)

The number of times an event occurs in “a single time period” at rate λ.

Geometric

The number of attempts before an event occurs, each event with probability p.

?????

The amount of time before an event occurs, each event at rate λ.

Define the distribution by $\Pr(X > t) = e^{-\lambda t}$. What is the CDF? The PDF?

CDF

$$F_X(x) =$$

$$F_X(t) = Pr(X \leq t) = 1 - Pr(X > t)$$

$$= 1 - e^{-\lambda t}$$

$$F_X(t) = \int_{-\infty}^{t} f_X(x)\,dx$$

$$f_X(t) = \frac{d}{dt}\left(F_X(t)\right) = \lambda e^{-\lambda t}$$

Exponential Distribution

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CDF

$$F_X(x) = \Pr(X \le t) = 1 - \Pr(X > t) = 1 - e^{-\lambda t}$$

PDF

Recall, $f_X(x) = \frac{d}{dx} F_X(x)$. So,

$$f_X(x) =$$

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CDF

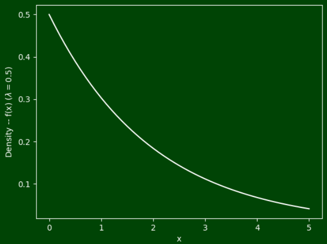
$$F_X(x) = \Pr(X \le t) = 1 - \Pr(X > t) = 1 - e^{-\lambda t}$$

PDF

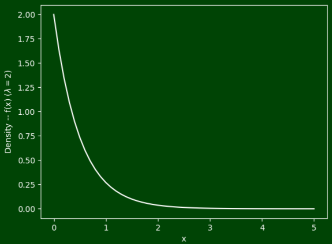
Recall, $f_X(x) = \frac{d}{dx} F_X(x)$. So,

$$f_X(x) = \frac{d}{dx}\left(1 - e^{-\lambda t}\right) = \lambda e^{-\lambda t}$$

Exponential$(1/2)$ PDF



Exponential(2) PDF



Expectation of Exponential

$$\mathbb{E}[X] = \int_0^\infty x f_X(x)\,dx$$

$$= \lambda \int_0^\infty x e^{-\lambda x}\,dx$$

$$u = x \qquad dv$$

$$du = dx \qquad v$$

$$f_X(t) = \lambda e^{-\lambda t}$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_{0}^{\infty} x e^{-\lambda x}\, dx$$

$$u = x \qquad dv = e^{-\lambda x}\, dx$$

$$du = dx \qquad v = \int e^{-\lambda x}\, dx = -\frac{1}{\lambda} e^{-\lambda x}$$

$$= \lambda \left(uv\big|_0^{\infty} - \int_0^{\infty} v\, du \right)$$

$$= -xe^{-\lambda x}\Big|_0^{\infty} - \lambda \int_0^{\infty} -\frac{1}{\lambda} e^{-\lambda x}\, dx$$

$$= -xe^{-\lambda x}\Big|_0^{\infty} - \frac{1}{\lambda} e^{-\lambda x}\Big|_0^{\infty}$$

$$= \left(-xe^{-\lambda x} - \frac{1}{\lambda} e^{-\lambda x} \right)\Big|_0^{\infty}$$

$$= \frac{1}{\lambda}$$

$$\mathbb{E}[X^2] =$$

$$\lambda \int_{-\infty}^{\infty} x^2 e^{-\lambda x}\, dx$$

$$u = x^2 \qquad dv =$$

$$du = 2x\, dx \qquad v =$$

$$= \lambda \left(uv\big|_0^{\infty} - \int_0^{\infty} v\, du \right)$$

$\mathbb{E}[X^2]$ of Exponential

$$
\begin{aligned}
\mathbb{E}\left[X^2\right] &= \int_{-\infty}^{\infty} x^2 \lambda e^{-\lambda x} \\
&= \lambda \int_0^{\infty} x^2 e^{-\lambda x}
\end{aligned}
$$

$$
\begin{aligned}
u &= x^2 & dv &= e^{-\lambda x}\, dx \\
du &= 2x\, dx & v &= \int e^{-\lambda x}\, dx = -\frac{1}{\lambda} e^{-\lambda x}
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda \left(uv\big|_0^{\infty} - \int_0^{\infty} v\, du \right) \\
&= -x^2 e^{-\lambda x}\Big|_0^{\infty} + \frac{2}{\lambda} \int_0^{\infty} x e^{-\lambda x} \\
&= -x^2 e^{-\lambda x}\Big|_0^{\infty} + 2\frac{\mathbb{E}[X]}{\lambda^2} \\
&= -x^2 e^{-\lambda x}\Big|_0^{\infty} + \frac{2}{\lambda^2} \\
&= \frac{2}{\lambda^2}
\end{aligned}
$$

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$$\mathbb{E}\left[X^2\right] = \frac{2}{\lambda^2}$$

$$\mathsf{Var}(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2 = \frac{1}{\lambda^2}$$

Definition (Memorylessness)

If $\Pr(X > s+t \mid X > s) = \Pr(X > t)$, we say X is "memoryless". In English, if you've waited s units of time, the probability that you wait t more units of time is exactly the same as if you'd just shown up.

The Geometric Distribution is Memoryless

What is the CDF of the geometric distribution? If $G \sim \text{Geometric}(p)$, then $\Pr(G > k) =$

$$\Pr(G > k) = p \sum_{i=k}^{\infty} (1-p)^i \; p = 1 - \Pr(G \le k) = (1-p)^k$$

$$\Pr(x > s+t \mid x > s) = \frac{\Pr(x > s+t \cap x > s)}{\Pr(x > s)} = \frac{(1-p)^{s+t}}{(1-p)^s} = (1-p)^t$$

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So, $\Pr(G > s+t \mid G > s) = \frac{\Pr(G>s+t \cap G>s)}{\Pr(G>s)} = \frac{\Pr(G>s+t)}{\Pr(G>s)} = \frac{(1-p)^{s+t}}{(1-p)^s} = (1-p)^t = \Pr(G > t)$.

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If $E \sim \text{Exponential}(\lambda)$, then $\Pr(E > x) = e^{-\lambda x}$.

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So,

$\Pr(E > s+t \mid E > s) = \frac{\Pr(E>s+t \cap E>s)}{\Pr(E>s)} = \frac{\Pr(E>s+t)}{\Pr(E>s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \Pr(E > t)$.

Which Distribution?

For each of the following r.v.'s, which distribution best fits the description?

- time until the next packet arrival at a server
- number of packet arrivals at a server in a minute
- number of packet arrivals at a server in an hour
- number of words until the next typo in a textbook
- time until a defective product is found if 1 product is inspected per second