CSE 312

Foundations of Computing II

Linearity of Expectation

Definition (Linearity of Expectation)

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$$\mathbb{E}[Z] = \mathbb{E}[X] + \mathbb{E}[Y]$$

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$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i]$$

$$\mathbb{E}[Z] = \sum_{k} k \Pr(Z=k) = \sum_{s \in \Omega} Z(s) \Pr(s)$$

$$X(s) + Y(s)$$

$$\sum X(s) \Pr(s) + \sum Y(s) \Pr(s)$$

$$\mathbb{E}[X] + \mathbb{E}[Y]$$

Sum of Two Dice Rolls

Consider the experiment:

```
A = RollDie(3)
B = RollDie(3)
C = A + B
```

What is $\mathbb{E}[C]$?

$$\mathbb{E}[C] = 2 \times \frac{1}{9} + 3 \times \frac{2}{9} + 4 \times \frac{3}{9} + 5 \times \frac{2}{9} + 6 \times \frac{1}{9} = 4$$

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What is $\mathbb{E}[C]$?

$$\mathbb{E}[C] = \mathbb{E}[A] + \mathbb{E}[B] = \frac{2}{3}(1 + 2 + 3) = \frac{12}{3} = 4$$

Binomial Random Variables

```
for i = 1 to n:
    if FlipCoin(p) == HEADS:
        X_i = 1
    else:
        X_i = 0
```

$$P_{X_i}(k) = \begin{cases} p & \text{if } k=1 \text{ (or HEADS)} \\ 1-p & \text{if } k=0 \text{ (or TAILS)} \end{cases}$$

$$Y = \sum X_i \qquad P_Y(k) =$$

$$\mathbb{E}[Y] = \qquad = np$$

$$P_Y(k) = \Pr(Y=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

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$$\mathbb{E}[Y] = \sum \mathbb{E}[X_i] = \sum_{i=1}^{n} p = np$$

$$\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1-p)$$

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$$\mathbb{E}[Y] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{k=0}^{n} \mathbb{E}[X_i] = np$$

There are n people at a party who have checked their hats in. At the end of the party, the hat-check clerk randomly gives them their hats back. What is the expected number of people who got their hat back?

Let X be the r.v. for the # of people who got their hat back.

Let X_i be the indicator r.v. for the ith person getting their hat back

$$\mathbb{E}[X] = \sum \mathbb{E}[X_i] = \sum_{i=1}^{n} 0(\ldots) + 1\left(\frac{1}{n}\right)$$

$$= 1$$

$$\frac{1}{n-i}$$

$$\mathbb{E}[XY]$$

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$\mathbb{E}[X_i] = 0\text{Pr}(X_i = 0) + 1\text{Pr}(X_i = 1) = \text{Pr}(X_i = 1) = \frac{1}{n}$.
So, $\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} 1/n = 1$ by Linearity of Expectation.

Linearity of Expectation Proof

Consider two r.v.'s: A and B. Then,

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$$\begin{aligned}
\mathbb{E}[A+B] &= \sum_{x\in\Omega} (A(x)+B(x))\mathsf{Pr}(x) \\
&= \sum_{x\in\Omega} A(x)\mathsf{Pr}(x) + \sum_{x\in\Omega} B(x)\mathsf{Pr}(x) \\
&= \mathbb{E}[A]+\mathbb{E}[B]
\end{aligned}$$