CSE 312

Foundations of Computing II

Welcome to CSE 312!

Outline

What Am I Getting Into?

Course Material

- ‘Combinatorics, Discrete Probability, Continuous Probability, Statistics/ML
- Computer Science applications and analyses
- Applications are what make this a CS course
- . . .

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CSE 311 vs. CSE 312

- Logic vs. Reasoning Under Uncertainty
- Proofs vs. Arguments

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- Logic vs. Reasoning Under Uncertainty
- Proofs vs. Arguments

Adam’s First Time Teaching This Course

- Please tell me if I’m going too fast or too slow.
- Please tell me if the homework is too hard or too easy.
- ...

Course Goals

During the course, we will. . .

- Extend the type of thinking learned in 311 to new situations
- Discover why combinatorial reasoning is useful to computer science
- Discover why probabilistic reasoning is useful to computer science

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After the course, you will be able to. . .

- Arm yourself in today's uncertain and biased world
- Rigorously analyze probabilistic algorithms

Support and Asking for Help

Resources

- Section every week!
- Lots of office hours!
- Piazza!

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Asking for help is not a sign of weakness; it's a sign of strength.

Boring Administrivia

Course Website

http://cs.uw.edu/312

Grading

- 50% homework, 20% midterm, 30% final

Textbook

Bertsekas and Tsitsiklis. Introduction to Probability.

Do what helps you most.

. . . but **active learning** has been proven to result in better performance.

Why Bother?

Why bother studying combinatorics and probability?
It's more math. . . we're still computer scientists. . .

Baseball Tournaments

Imagine you're designing a tournament for n little-league baseball teams. There are several different ways that they could play each other:

- Each team plays every other team once. (Round Robin)
- Each team plays until they lose. (Single Elimination)
- Each team plays until they lose twice. (Double Elimination)

You have been tasked with figuring out which type of tournament is best for the children to play in. Since each game costs your boss money, he would like them to play a minimal number of games. Which type of tournament should you recommend?

DNA Sequencing

Imagine you're working in bioinformatics, and you've been asked to identify if a strand of DNA could have replicated from from a set of other strands of DNA. Recall that DNA strands are just strings of $\{A, C, T, G\}$.

Your first thought is to write a program to brute force all the possibilities. Is this a reasonable approach?

Poker

You're playing a game of poker and you have a pair of 10's and a pair of queens.

How likely are you to win?

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4. **Optimization.** What is the **best** solution to a problem? Why can't we do better?

It’s about the process

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   How do computers maintain data integrity?
4. **Optimization.** What is the **best** solution to a problem? Why can’t we do better?
   How does a GPS know the best route between any two locations?

To solve each of these questions, you have to reason about **how many** of something there are. This process is “thinking combinatorially”, and we’re going to talk about it for the next **two weeks**!

Thinking combinatorially can sometimes make very difficult problems much easier.

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Let's build up a "toolbox" of approaches we can take!

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This may seem a little strange, but our three most powerful tools in counting are laws of **sets**!

Definition (Rule of Product)

If we have sets $X_1, X_2, \ldots X_n$ then

$$|X_1 \times X_2 \times \cdots \times X_n| = |X_1| \times |X_2| \times \cdots \times |X_n|$$

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The outcomes of rolling two six-sided dice are members of $D_6 \times D_6$.

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So, the number of outcomes is $6 \times 6 = 36$. □

UGH! Do we have to write that every time?

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Example

How many ways can I roll two six-sided dice?

Proof.

We know that there are six ways to roll a single die. To roll two dice, we follow this procedure:

- Roll one die.
- Roll one die.

Each step of the procedure has six possibilities; so, multiplying them together by the Rule of Product, we get $6 \times 6 = 36$ outcomes. □

Definition (Disjoint Sets)

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$$\forall (i \neq j).\, X_i \cap X_j = \varnothing$$

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How many ways can I roll two six-sided dice to get a sum of 4?

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Proof.

Note that the first roll could be 1 through 6.

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How many ways can I roll two six-sided dice to get a sum of 4?

Proof.

Note that the first roll could be 1 through 6. We partition on these cases:

- If the first roll is a 1, then the second roll must be 3.

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- If the first roll is a 2, then the second roll must be 2.
- If the first roll is a 3, then the second roll must be 1.
- If the first roll is 4, 5, or 6, then we can never sum to 4.

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Note that these cases are mutually exclusive.

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Note that these cases are mutually exclusive. Furthermore, this covers all the possible cases for the first die.

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{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)

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- If the first roll is 4, 5, or 6, then we can never sum to 4.

Note that these cases are mutually exclusive. Furthermore, this covers all the possible cases for the first die. Putting these together, we see that $1+1+1+0+0+0 = 3$ is our answer by the Rule of Sum. □

Counting by Complement

Sometimes, instead of counting the things we want, we count the things we **don't** want and remove them.

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If $\mathcal{U}$ is the universal set, then

$$A = \mathcal{U} \setminus \overline{A}$$

$$\{1,2,3\} \setminus \{3\} = \{1,2\}$$

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- Choose the nth bit.

Since each step of this procedure has 2 options, the total number of binary strings of length n is $\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} = 2^n$ by the Rule of Product.

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Now, we count how many binary strings of length n have no 1's.

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Now, we count how many binary strings of length n have no 1's. We use the same procedure as before, except, now, we only have 1 choice at each step.

Example

How many binary strings of length n are there that have at least one 1.

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Now, we count how many binary strings of length n have no 1's. We use the same procedure as before, except, now, we only have 1 choice at each step. It follows that there is 1 bad binary string.

Example

How many binary strings of length n are there that have at least one 1.

Proof.

First, we show that there are 2^n binary strings. To generate a binary string, we use an n-step process:

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Now, we count how many binary strings of length n have no 1's. We use the same procedure as before, except, now, we only have 1 choice at each step. It follows that there is 1 bad binary string.
So, Counting by Complement, we see that there are $2^n - 1$ binary strings with at least one 1. □

Now that we know what we're trying to do, let's build up the primitives of our language.

Think of these like **if statements** and **for loops** in programming.

We can use these to build up larger, more complicated counting arguments!

Factorials

Primitive: Arranging $\{x_1, x_2, \ldots, x_n\}$

We would like to arrange n distinct things, $\{x_1, x_2, \ldots, x_n\}$, in a row:

How many places could we put x_1?

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□ □ □ ··· □ □ □

How many places could we put x_1? n

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How many places could we put x_2?

Primitive: Arranging $\{x_1, x_2, \ldots, x_n\}$

We would like to arrange n distinct things, $\{x_1, x_2, \ldots, x_n\}$, in a row:

How many places could we put x_1? n

How many places could we put x_2? $n - 1$

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We would like to arrange n distinct things, $\{x_1, x_2, \ldots, x_n\}$, in a row:

$$\boxed{}\quad \boxed{x_2}\quad \boxed{x_3}\quad \cdots\quad \boxed{x_1}\quad \boxed{}\quad \boxed{}$$

How many places could we put x_1? n

How many places could we put x_2? $n-1$

$\ldots$

How many places could we put x_k?

Factorials

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How many places could we put x_1? n

How many places could we put x_2? $n-1$

$\ldots$

How many places could we put x_k? $n-(k-1)$

Primitive: Arranging $\{x_1, x_2, \ldots, x_n\}$

We would like to arrange n distinct things, $\{x_1, x_2, \ldots, x_n\}$, in a row:

$$\boxed{x_{n-1}} \quad \boxed{x_2} \quad \boxed{x_3} \quad \cdots \quad \boxed{x_1} \quad \boxed{} \quad \boxed{x_k}$$

How many places could we put x_1? n
How many places could we put x_2? $n-1$
...
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How many places could we put x_n?

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Proof.

We can arrange $\{x_1, x_2, \ldots, x_n\}$ in an n-step process, where, on step k, we place x_k. There are $n-(k-1)$ ways to do step k, since there are that many spots remaining. It follows that the number of ways to arrange our set is $n(n-1)\cdots 2(1) = n!$ by Rule of Product. $\square$

Primitive: Choosing a subset of k elements of $\{x_1, x_2, \ldots, x_n\}$

$$\binom{n}{k}$$

For now, we don't care how to calculate this explicitly. Treat it as a primitive, just like you would $\sin$ in a calculus class.

Problems

$\{1, 2, 3\} \rightarrow \{1\}, \{2\}, \{3\}$

$\binom{3}{1}$

Quick re-cap:

A, C, T, G exactly 4 C's

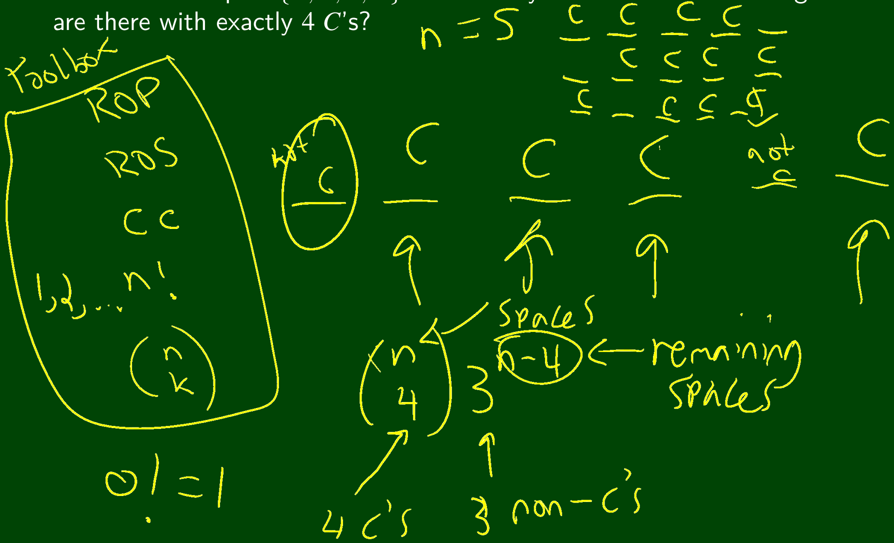
Toolbox

- Rule of Product: To calculate how many outcomes there are of a multi-step procedure, multiply the numbers together.
- Rule of Sum: If we have a counting argument that enumerates **disjoint** cases, we can add the numbers together.
- Counting By Complement: Sometimes, it's easier to (1) count the total, (2) count the number of things that **don't** satisfy the property

Primitives

- $n!$: The number of ways to order n **distinct** items
- $\binom{n}{k}$: The number of ways to choose k of n **distinct** items.

DNA is made up of $\{A, C, T, G\}$. How many strands of DNA of length n are there with exactly 4 C's?



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The number of ways to do the first step is $\binom{n}{4}$, and the number of ways to do the other $n-4$ steps is 3. Using the Rule of Product, we get that there are $\binom{n}{4}3^{n-4}$ possible strands of DNA with 4 C's. □

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We count the hands with the following process:

- Choose three of the four Aces.
- Out of the remaining 49 cards, choose 2 of them.

By the Rule of Product, the number of five card hands with three or four Aces is $\binom{4}{3}\binom{49}{2}$.

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Our argument overcounts! If a counting argument is correct, we must be able to trace an output to a particular choice pattern.

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We partition on if there are three Aces or four.

$\binom{4}{3}\binom{5}{3} 48 \cdot 47 \cdot 3!$ order aces $= \binom{4}{3}\binom{48}{2}$

ordered

$+ \binom{4}{4}\binom{48}{1}$

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- If there are three Aces, choose which Aces there are, and then choose two non-Aces.

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We partition on if there are three Aces or four.

- If there are three Aces, choose which Aces there are, and then choose two non-Aces. By Rule of Product, this works out to $\binom{4}{3}\binom{48}{2}$.
- If there are four Aces, choose all four Aces, and then choose the remaining card.

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Note that every hand with 3 or 4 Aces must either have 3 or 4 Aces, and that no hand can have both 3 and 4 Aces; so, these cases form a partition.

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- If there are three Aces, choose which Aces there are, and then choose two non-Aces. By Rule of Product, this works out to $\binom{4}{3}\binom{48}{2}$.
- If there are four Aces, choose all four Aces, and then choose the remaining card. By Rule of Product, this works out to $\binom{4}{4}\binom{48}{1}$.

Note that every hand with 3 or 4 Aces must either have 3 or 4 Aces, and that no hand can have both 3 and 4 Aces; so, these cases form a partition.It follows, by Rule of Sum, that the number of five card hands with three or four Aces is $\binom{4}{3}\binom{48}{2}+\binom{4}{4}\binom{48}{1}$. □

Today's Takeaways!

- Hopefully you're excited!

- Why do we care about counting things?