CSE 312

Foundations of Computing II

Independence

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- $\Pr(C_P) = 1$
- $\Pr(C_S) = \frac{1}{8}$
- $\Pr(C_P \cap C_S) = \frac{1}{8}$

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What does this mean?

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Tempting Answer: Pete and Sandy's decisions to go to class aren't reliant on each other.

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What does this mean?

Tempting Answer: Pete and Sandy's decisions to go to class aren't reliant on each other.

Actual Answer: There are a total of 8 sessions of CSE 000. Sandy met Pete at the first class which they both attended. For all future sessions, Pete took notes for Sandy, and she never showed up again.

Consider the following experiment:

```
die1 = RollDie(6)
die2 = RollDie(6)
```

Let D_1 be the event die1 $= 1$.
Let D_2 be the event die2 $= 1$.
Let S_5 be the event die1 $+$ die2 $= 5$.

Are D_1 and D_2 dependent or independent?

$\Pr(D_1) = \frac{1}{6}$

$\Pr(D_2) = \frac{1}{6}$

$\Pr(D_1 \cap D_2) = \frac{1}{36} = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$

Are D_1 and S_5 dependent or independent?

$\Pr(D_1) = \frac{1}{6}$

$\Pr(S_5) = \frac{1}{9}$

$\Pr(D_1 \cap S_5) = \frac{1}{36}$

$|S_5| = |\{(1,4), (2,3), (3,2), (4,1)\}|$

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Let D_1 be the event $\texttt{die1} = 1$.
Let D_2 be the event $\texttt{die2} = 1$.
Let S_5 be the event $\texttt{die1} + \texttt{die2} = 5$.

Are D_1 and D_2 dependent or independent?

$\Pr(D_1) = \Pr(D_2) = \frac{1}{6}$; $\Pr(D_1 \cap D_2) = \frac{1}{36}$. Note that $\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$. So, D_1 and D_2 are independent.

Are D_1 and S_5 dependent or independent?

Rolling Dice

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$\Pr(D_1) = \Pr(D_2) = \frac{1}{6}$; $\Pr(D_1 \cap D_2) = \frac{1}{36}$. Note that $\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$. So, D_1 and D_2 are independent.

Are D_1 and S_5 dependent or independent?

Note that $|S_5| = |\{(1,4),(2,3),(3,2),(4,1)\}| = 4$. So, $\Pr(S_5) = \frac{4}{36} = \frac{1}{9}$, but $\Pr(D_1 \cap S_5) = \frac{1}{36}$. So, D_1 and S_5 are **dependent** events.

Fair Coins

Consider the experiment where we flip n fair coins.

- What is the probability that we get n HEADs?

  $|E| = 1^n = 1$

  $|\Omega| = 2^n$

  $\left(\frac{1}{2}\right)^n$

- What is the probability that the first $0 \le k \le n$ flips are HEADs and the remaining ones are tails?

  $|E| = 1^n$

  $|\Omega| = 2^n$

  $\left(\frac{1}{2}\right)^n$

- What is the probability that we get $0 \le k \le n$ HEADs overall?

  $|E| = \binom{n}{k}$

  $|\Omega| = 2^n$

Consider the experiment where we flip n fair coins.

- What is the probability that we get n `HEAD`s?

$$\frac{1^n}{2^n}$$

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Note that $\sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k} = \left(\frac{1}{2}\right)^n 2^n = 1.$

General Independence

In general, events $E_1, E_2, \ldots, E_n$ are independent iff for **every subset** $S \subseteq [n]$, we have:

$$\Pr\left(\bigcap_{i \in S} E_i\right) = \prod_{i \in S} \Pr(E_i)$$

That is, if we have E_1, E_2, E_3, $\{E_1, E_2\}, \{E_1, E_3\}, \{E_2, E_3\}, \{E_1, E_2, E_3\}$ all have to be independent to consider E_1, E_2, and E_3 independent.

Example

```
a = FlipCoin(1/2)
if a == HEADS:
   b = RollDie(6)
   c = RollDie(9)
else:
   b = RollDie(6)
   c = RollDie(4)
```

Let A be the event $\mathtt{a} = \mathtt{HEADS}$.
Let B be the event $\mathtt{b} = 1$.
Let C be the event $\mathtt{c} = 1$.
Are A, B, and C independent? What subsets of them are independent?

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```

$$\Pr(A|B) = \frac{\Pr(A\cap B)}{\Pr(B)}$$

$$\Pr(A\cap B) = \Pr(A|B)\Pr(B)$$

$$\Pr(A) = \Pr(A|B)\Pr(B) + \Pr(A|\bar{B})\Pr(\bar{B})$$

Let A be the event $\mathsf{a} = \mathtt{HEADS}$.
Let B be the event $\mathsf{b} = 1$.
Let C be the event $\mathsf{c} = 1$.
Are A, B, and C independent? What subsets of them are independent?

$\Pr(A) = \frac{1}{2}$

$\Pr(A\cap B) = \Pr(B\cap A) = \Pr(B|A)\Pr(A)$

$\Pr(B) = \Pr(B|A)\Pr(A) + \Pr(B|\bar{A})\Pr(\bar{A})$

$= \frac{1}{6}\frac{1}{2} + \frac{1}{6}\frac{1}{2} = \frac{1}{6}$

Independent → related or → not related

not related → independent

$\Pr(C) = 13/72$ $\quad \Pr(A\cap C) = \frac{1}{18}$ $\quad \Pr(B\cap C) = \frac{1}{36}$

$\frac{1}{78}, \frac{1}{36}, \frac{13}{216}, \frac{1}{54} + \frac{1}{24}, \frac{13}{432}$

$\Pr(B\cap C) = \Pr(B\cap C|A)\Pr(A) + \Pr(B\cap C|\bar{A})\Pr(\bar{A})$

$\frac{1}{54} \quad \frac{1}{2} \quad + \quad \frac{1}{24} \quad \frac{1}{2} = \frac{13}{432}$

Biased Coins

Consider the experiment where we flip n **independent** coins with bias p ($\Pr(\mathtt{HEADS}) = p$).

- What is the probability that we get n HEADs?

  p^n

- What is the probability that the first $0 \le k \le n$ flips are HEADs and the remaining ones are tails?

  $p^k (1-p)^{n-k}$

- What is the probability that we get $0 \le k \le n$ HEADs overall?

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- What is the probability that we get $0 \le k \le n$ HEADs overall?

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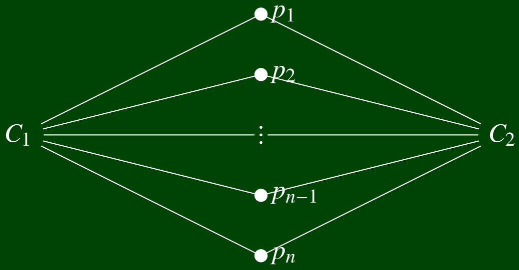
Note that $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1^n = 1$.

Parallel Routers

Suppose there are n routers in parallel and the ith router fails independently with probability p_i.



What is the probability that C_1 can communicate with C_2?

$$\Pr(C_1 \text{ comm. } C_2) = 1 - \Pr(\text{all fail}) = 1 - \prod_{i=1}^{n} p_i$$

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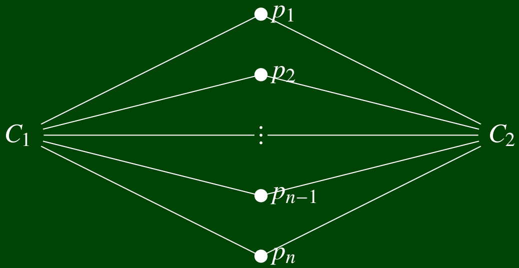

What is the probability that C_1 can communicate with C_2?

$\Pr(C_1 \text{ communicates with } C_2) = 1 - \Pr(\text{all routers fail}) = 1 - p_1 p_2 \cdots p_n$

Series Routers

Suppose there are n routers in series and the ith router fails indepdendently with probability p_i.

What is the probability that C_1 can communicate with C_2?

$\Pr(C_1$ communicates with $C_2) = \Pr($no router fails$) =$
$(1-p_1)(1-p_2)\cdots(1-p_n)$

If $\Pr(F) > 0$, then E and F are independent iff $\Pr(E \mid F) = \Pr(E)$

Proof.

If $\Pr(F) > 0$, then E and F are independent iff $\Pr(E \mid F) = \Pr(E)$

Proof.

($\Rightarrow$) Since E and F are independent, $\Pr(E \cap F) = \Pr(E)\Pr(F)$. Then, using the definition of conditional probability,
$\Pr(E)\Pr(F) = \Pr(E \mid F)\Pr(F)$. So, $\Pr(E \mid F) = \Pr(E)$.

($\Leftarrow$) Since $\Pr(E \mid F) = \Pr(E)$, we can apply the definition of conditional probability to get $\frac{\Pr(E \cap F)}{\Pr(F)} = \Pr(E)$ (since $\Pr(F) \neq 0$). Then, we have $\Pr(E \cap F) = \Pr(F)\Pr(E)$. □