CSE 312

Foundations of Computing II

Fancy Counting

$$1, 2, 3$$

Many of the questions we ask in counting are instances of the question:

How many ways are there to place n balls into m bins?

	n **Indistinguishable Balls** m **Distinguishable Bins**	n **Distinguishable Balls** m **Distinguishable Bins**
Exactly One		
At Most One		Together
At Least One	HW	Together
Any Number	Together	

	n **Indistinguishable Balls** m **Distinguishable Bins**	n **Distinguishable Balls** m **Distinguishable Bins**
Exactly One	1 if $n = m$ else 0	
At Most One		Together
At Least One	HW	Together
Any Number	Together	

How many ways can the balls match up with the bins?

	n **Indistinguishable Balls** m **Distinguishable Bins**	n **Distinguishable Balls** m **Distinguishable Bins**
Exactly One	1 if $n = m$ else 0	$n!$ if $n = m$ else 0
At Most One		Together
At Least One	HW	Together
Any Number	Together	

How many ways there are to rearrange the balls to match up with the bins?

	n **Indistinguishable Balls** m **Distinguishable Bins**	n **Distinguishable Balls** m **Distinguishable Bins**
Exactly One	1 if $n = m$ else 0	$n!$ if $n = m$ else 0
At Most One	$\binom{m}{n}$	Together
At Least One	HW	Together
Any Number	Together	

Since the balls are indistinguishable, we're just choosing which bins to put them in.

	n **Indistinguishable Balls** m **Distinguishable Bins**	n **Distinguishable Balls** m **Distinguishable Bins**
Exactly One	1 if $n = m$ else 0	$n!$ if $n = m$ else 0
At Most One	$\binom{m}{n}$	Together
At Least One	HW	Together
Any Number	Together	m^n

For each ball, choose a bin (with replacement)

	n **Indistinguishable Balls** m **Distinguishable Bins**	n **Distinguishable Balls** m **Distinguishable Bins**
Exactly One	1 if $n = m$ else 0	$n!$ if $n = m$ else 0
At Most One	$\binom{m}{n}$	
At Least One	HW	Together
Any Number	Together	m^n

	n **Indistinguishable Balls** m **Distinguishable Bins**	n **Distinguishable Balls** m **Distinguishable Bins**
Exactly One	1 if $n = m$ else 0	$n!$ if $n = m$ else 0
At Most One	$\binom{m}{n}$	$\binom{m}{n} n!$
At Least One	HW	Together
Any Number	Together	m^n

Choose which bins get a ball, then order the balls

	n **Indistinguishable Balls** m **Distinguishable Bins**	n **Distinguishable Balls** m **Distinguishable Bins**
Exactly One	1 if $n = m$ else 0	$n!$ if $n = m$ else 0
At Most One	$\binom{m}{n}$	$\binom{m}{n}n!$
At Least One	HW	Together
Any Number		m^n

How many ways are there to place n **indistinguishable** balls into m **distinguishable** bins?

Start out with $n+m-1$ indistinguishable $\circ$'s:

$$\underbrace{\circ\circ\circ\circ\cdots\circ\circ\circ\circ}_{n+m-1 \text{ of these}}$$

Choose $m-1$ of the $\circ$'s to turn into "dividers":

$$\underbrace{\circ\circ\,|\,|\,\circ\cdots\circ\,|\,\circ\circ}_{n\ \circ\text{'s},\ m-1 \text{ dividers}}$$

This means that there are n balls and $m-1$ dividers (which makes m bins!). The only step in our counting argument was to choose $m-1$ of the $n+m-1$ $\circ$'s to be dividers. So, there are $\binom{n+m-1}{m-1}$ ways to place these balls into bins.

	n **Indistinguishable Balls** m **Distinguishable Bins**	n **Distinguishable Balls** m **Distinguishable Bins**
Exactly One	1 if $n = m$ else 0	$m!$ if $n = m$ else 0
At Most One	$\binom{m}{n}$	$\binom{m}{n}m!$
At Least One	HW	Together
Any Number	$\binom{n+m-1}{m-1}$	m^n

	n **Indistinguishable Balls** m **Distinguishable Bins**	n **Distinguishable Balls** m **Distinguishable Bins**
Exactly One	1 if $n = m$ else 0	$m!$ if $n = m$ else 0
At Most One	$\binom{m}{n}$	$\binom{m}{n}m!$
At Least One	HW	
Any Number	$\binom{n+m-1}{m-1}$	m^n

Remember how we started counting with set laws? Well, there's one more. . .

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

This is called inclusion-exclusion, and it's useful for when you want to compute the size of a union that **isn't disjoint**.
Going to three sets, we have:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) + |A_1 \cap A_2 \cap A_3|$$

More generally, Inclusion-Exclusion says:

singles - doubles + triples - quads + . . .

The next obvious question is "how do I use this?". The big thing to get about inclusion-exclusion is how to define the A_i's.
Let's do an example.

How many ways are there to choose three numbers from three sets (A, B, and C) such that:

- $A = \{1,2,3,\ldots n\}$
- $B = \{1,2,3,\ldots,m\}$
- $C = \{1,2,3,\ldots,\ell\}$
- At least one of the numbers chosen is a 1.

The reason we should think inclusion-exclusion is that the problem specifies "at least" about something related to all the sets.

There are three sets here; so, it's likely that we'll be using inclusion-exclusion with three A_i's:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) + |A_1 \cap A_2 \cap A_3|$$

Before we do anything else, we need to determine what A_i is supposed to be. If we have defined things correctly, then the set we're looking for should be $A_1 \cup A_2 \cup A_3$.

Triple the Example, Triple the Fun

How many ways are there to choose three numbers from three sets (A, B, and C) such that:

- $A = \{1, 2, 3, \ldots n\}$
- $B = \{1, 2, 3, \ldots, m\}$
- $C = \{1, 2, 3, \ldots, \ell\}$
- At least one of the numbers chosen is a 1.

We want to use:

A_i is the set of triples with the ith coordinate equal to 1.

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) + |A_1 \cap A_2 \cap A_3|$$

$$|A_1 \cup A_2 \cup A_3| = m\ell + n\ell + nm - (\ell + m + n) + 1$$

Back to our problem. . .

How many ways are there to place n **distinguishable** balls into m **distinguishable** bins such that every bin gets **at least one ball**?

Let's count by complement:

How many ways are there to place n **distinguishable** balls into m **distinguishable** bins such that some bin gets **no balls**?

What are our A_i's?

A_i is the set of outcomes with no balls in the ith bin.

How many ways are there to place n **distinguishable** balls into m **distinguishable** bins such that some bin gets **no balls**?

A_i is the set of outcomes with no balls in the ith bin.

Inclusion-exclusion works by counting the sizes of the various intersections.

Consider the intersection of k distinct A_i's. That is, how many ways are there to place the balls in the bins such that some bin gets no balls AND each of the k A_i's gets no balls?

What is the cardinality of this set?

Well, regardless of which A_i's, we already have at least one bin with no balls. So, all we have to do is assign every ball to one of the other bins.

There are $(m-k)^n$ ways to do this.

Back to our problem... (continued)

How many ways are there to place n **distinguishable** balls into m **distinguishable** bins such that some bin gets **no balls**?

A_i is the set of outcomes with no balls in the ith bin.

$$|A_{x_1} \cap A_{x_2} \cap \cdots \cap A_{x_k}| = (m-k)^n$$

How many ways are there to choose k of the A_i's?

There are m A_i's; so, to choose k of them... $\binom{m}{k}$.

Putting it all together...

$$\binom{m}{1}(m-1)^n - \binom{m}{2}(m-2)^n + \cdots + (-1)^{k+1}\binom{m}{k}(m-k)^n + \cdots + \binom{m}{m}(m-m)^n$$

So, we get $\sum_{i=1}^{k}(-1)^{k+1}\binom{m}{k}(m-k)^n$.

Finally, recall that we are **counting by complement**. So, we subtract our answer from m^n.

	n **Indistinguishable Balls** m **Distinguishable Bins**	n **Distinguishable Balls** m **Distinguishable Bins**
Exactly One	1 if $n = m$ else 0	$m!$ if $n = m$ else 0
At Most One	$\binom{m}{n}$	$\binom{m}{n} m!$
At Least One	HW	$\sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$
Any Number	$\binom{n+m-1}{m-1}$	m^n

Pigeonhole Principle

If you have n pigeons and m holes where $m < n$, then two pigeons must be in the same hole.

If you're at a party and some number of the n people shake hands, must two of them have shaken the same number of hands?

Solution

Note that someone either shakes hands with everyone or not. If someone does, then nobody shakes hands 0 times; if somebody does not, then nobody shakes hands n times. Either way, there are only $n-1$ options for n people; so, by the pigeonhole principle, two people shake the same number of hands.

Pigeonhole Principle

If you have n pigeons and m holes where $m < n$, then two pigeons must be in the same hole.

If you have six people who are either mutual friends or mutual strangers, prove there is a group of 3 strangers or 3 friends.

Solution

Consider one of the people, Sally. She has five "connections" to other people; so, by the pigeonhole principle, 3 of them must be strangers or friends. Without loss of generality, suppose they're friends with Sally. Consider the connections between pairs of those friends. If any of them are friends, we're done. Otherwise, they're all strangers, and we're done.