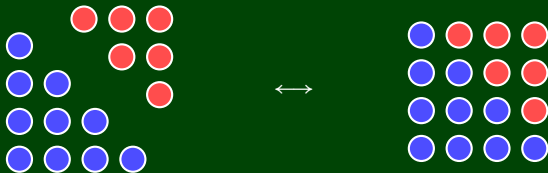


CSE 312

Foundations of Computing II

Counting in Two Ways



- Extra practice section in EEB 037 from 3:30pm - 4:30pm on Thursday. You don't need to sign up; just show up. Yes, the problems will be posted on the website.
- HW Grading
- Adam's Open Door Policy
- HW 1 Out

Outline

1 What is Counting in Two Ways?

2 Examples

3 Binomial Theorem

4 More Examples

Definition (Counting in Two Ways)

If we have a set X and natural numbers n, m , then if $n = |X|$ and $m = |X|$, then $n = |X| = m$.

Okay, duh, but...

Definition (Triangle Numbers)

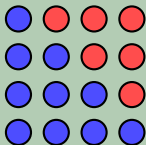
The n th Triangle Number, $\Delta_n = 1 + 2 + \dots + n$.

Let's prove that $n^2 = \Delta_n + \Delta_{n-1}$.

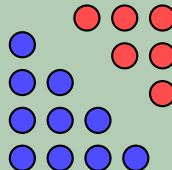
Proof 1: Induction

We did enough of that in CSE 311.

Proof 2: Counting in Two Ways



Make a square with n dots on each side.



Make a square by combining a triangle of height n and a triangle of height $n-1$.

Outline

1 What is Counting in Two Ways?

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Prove that $\binom{n}{k} = \binom{n}{n-k}$.

Set

We claim that both sides of this identity count the number of **committees of size k we can make out of n possible members.**

Way 1: Use the definition

Note that the left side counts this by definition.

Way 2: Be Clever!

The right side chooses $n-k$ **people to be excluded from the committee.** This leaves behind k to be included.

How many ways are there to arrange n people in a row?

Way 1: Use $n!$

Just order them: there are $n!$ ways to do this.

Way 2: Use $\binom{n}{k}$

Choose the first k people in the row. Then, order them. Then, order the remaining people.

There are $\binom{n}{k}$ ways to do the first step, $k!$ ways to do the second step, and $(n-k)!$ ways to do the third step. By the rule of product, there are $\binom{n}{k}k!(n-k)!$ ways to order n people.

The Point. . .

We've just shown that $n! = \binom{n}{k}k!(n-k)!$; that is: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

The first question to answer is what set both sides are counting.

- If one of the sides is $\binom{n}{k}$, we should think. . .
Committees of size k from n people.
- If one of the sides is 312^k , we should think. . .
Strings of length k with 312 possibilities for each character.
- If one of the sides is $n!$, we should think. . .
Arrangments of n distinct items.

How many ways are there to arrange n people in a circle?

We can pull the same trick. Temporarily denote this number as C_n . Let's answer the same question as before, but this time using C_n :

How many ways are there to arrange n doggies in a row?

Way 1: Use $n!$

Just order them: there are $n!$ ways to do this.

Way 2: Use C_n

First, place the doggies in a circle. (There are C_n ways to do this)
Then, split the circle open by choosing a single doggie to lead the line.
(There are n ways to do this)

The Point...

We've just shown that $n! = C_n n$; that is: $C_n = \frac{n!}{n} = (n-1)!$

Prove

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

by counting in two ways.

Both Sides Count...

The number of ways to choose a group of k doggies from n doggies.

Way 1: Use $\binom{n}{k}$

Just choose them: there are $\binom{n}{k}$ ways to do this.

Way 2: Get Clever!

Partition the choices into two sets: (1) groups without **Hopper**, (2) groups with **Hopper**.

There are $\binom{n-1}{k}$ ways to choose a group that DOES NOT have **Hopper**.

There are $\binom{n-1}{k-1}$ ways to choose a group that DOES have **Hopper**.

How many ways are there to re-arrange the letters in the word APPLE?
Call this number N .

Both Sides Count. . .

The number of ways to rearrange AP_1P_2LE .

Way 1: Use $n!$

Just arrange them: there are $n!$ ways to do this.

Way 2: Get Clever!

Arrange the letters of APPLE (there are N ways to do this). Arrange the order of the P's (there are $2!$ ways to do this).

So, $n! = N \times 2!$. So, $N = \frac{n!}{2!}$