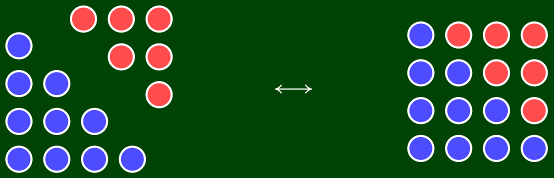
CSE 312

Foundations of Computing II

Counting in Two Ways



- Extra practice section in EEB 037 from 3:30pm - 4:30pm on Thursday. You don't need to sign up; just show up. Yes, the problems will be posted on the website.

- HW Grading

- Adam's Open Door Policy

- HW 1 Out

Outline

1. What is Counting in Two Ways?
2. Examples
3. Binomial Theorem
4. More Examples

Definition (Counting in Two Ways)

If we have a set X and natural numbers n, m, then if $n = |X|$ and $m = |X|$, then $n = |X| = m$.

Okay, duh, but...

$$|X| = n$$

$$|X| = m$$

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Let's prove that $n^2 = \triangle_n + \triangle_{n-1}$.

$|S| = n^2$

$|S| = \Delta_n + \Delta_{n-1}$

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Proof 1: Induction

We did enough of that in CSE 311.

Triangles and Squares

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Proof 1: Induction

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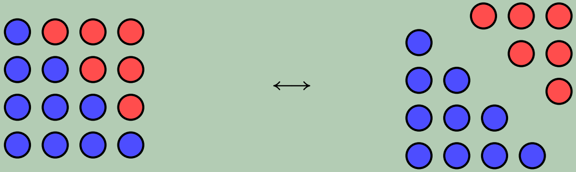
Proof 2: Counting in Two Ways



Make a square with n dots on each side.

Make a square by combining a triangle of height n and a triangle of height $n-1$.

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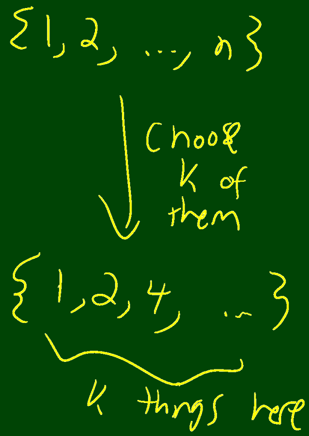
Prove that $\binom{n}{k} = \binom{n}{n-k}$.



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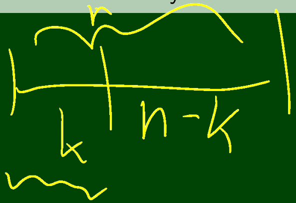

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We claim that both sides of this identity count the number of **committees of size k we can make out of n possible members**.

Way 1: Use the definition

Note that the left side counts this by definition.

Way 2: Be Clever!

The right side chooses $n-k$ **people to be excluded from the committee**. This leaves behind k to be included.

How many ways are there to arrange n people in a row?

$$n! = \binom{n}{k} \cdot \left[k! \, (n-k)! \right]$$

{A, B, C}

end with: ordered people

choose the first k people

start with: set of distinct people

{A, B, C, D, E}

$\binom{n-k}{n-k}$

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Way 2: Use $\binom{n}{k}$

Choose the first k people in the row. Then, order them. Then, order the remaining people.

There are $\binom{n}{k}$ ways to do the first step, $k!$ ways to do the second step, and $(n-k)!$ ways to do the third step. By the rule of product, there are $\binom{n}{k}k!(n-k)!$ ways to order n people.

How many ways are there to arrange n people in a row?

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The Point...

We've just shown that $n! = \binom{n}{k}k!(n-k)!$; that is: $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

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- If one of the sides is $\binom{n}{k}$, we should think...

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- If one of the sides is 312^k, we should think...
  Strings of length k with 312 possibilities for each character.
- If one of the sides is $n!$, we should think...
  Arrangments of n distinct items.

How many ways are there to arrange n people in a circle?

Set: arrange n doggies in a row

$$n! = C_n \cdot [\ n\ ]$$

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We can pull the same trick. Temporarily denote this number as C_n. Let's answer the same question as before, but this time using C_n:

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How many ways are there to arrange n doggies in a row?

Way 1: Use $n!$

Just order them: there are $n!$ ways to do this.

Way 2: Use C_n

First, place the doggies in a circle. (There are C_n ways to do this)
Then, split the circle open by choosing a single doggie to lead the line. (There are n ways to do this)

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The Point...

We've just shown that $n! = C_n n$; that is: $C_n = \frac{n!}{n} = (n-1)!$

Prove

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

by counting in two ways.

means use rule of sum

Set: Committee of k doggies chosen from n doggies one of whom is Hopper

Left: by def'n

Right:

And Another...

Prove

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Both Sides Count...

The number of ways to choose a group of k doggies from n doggies.

And Another...

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Both Sides Count...

The number of ways to choose a group of k doggies from n doggies.

Way 1: Use $\binom{n}{k}$

Just choose them: there are $\binom{n}{k}$ ways to do this.

And Another. . .

Prove

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

by counting in two ways.

Both Sides Count. . .

The number of ways to choose a group of k doggies from n doggies.

Way 1: Use $\binom{n}{k}$

Just choose them: there are $\binom{n}{k}$ ways to do this.

Way 2: Get Clever!

Partition the choices into two sets: (1) groups without **Hopper**, (2) groups with **Hopper**.

There are $\binom{n-1}{k}$ ways to choose a group that DOES NOT have **Hopper**.

There are $\binom{n-1}{k-1}$ ways to choose a group that DOES have **Hopper**.

How many ways are there to re-arrange the letters in the word APPLE? Call this number N.

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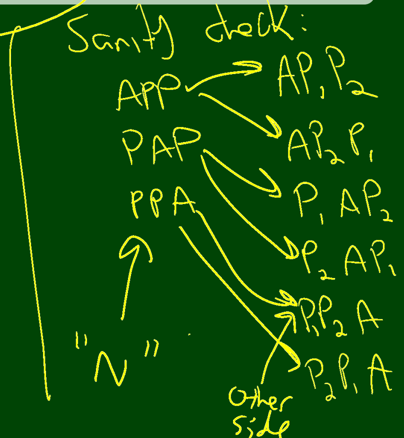
The number of ways to rearrange $\mathrm{AP_1P_2LE}$.

$5! = N.$



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Way 2: Get Clever!

Arrange the letters of APPLE (there are N ways to do this). Arrange the order of the P's (there are $2!$ ways to do this).

So, $n! = N \times 2!$. So, $N = \dfrac{n!}{2!}$