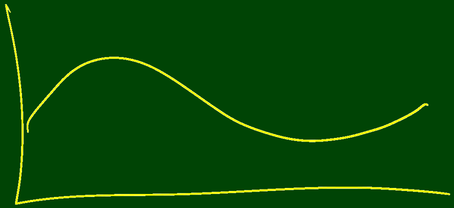
CSE 312

Foundations of Computing II

Continuous Random Variables

(aka Calculus time)



Discrete vs. Continuous

Until now, all r.v.'s have taken on **discrete** values (elements of $\mathbb{N}$); now, we'll consider **continuous** values (elements of $\mathbb{R}$).

Discrete	Continuous
$X \in [6]$	$x \in (0,6)$

$$p_X(k) = \begin{cases} \frac{1}{6} & \text{if } k \in [6] \\ 0 & \text{otherwise} \end{cases}$$

$$0 \le p_X(k) \le 1 \longrightarrow f_X(x) \ge 0$$

$$\sum_k p_X(k) = 1 \qquad \int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

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probability mass function	probability density function
$p_X(x)$	$f_X(x)$

Definition (Probability Density Function (PDF))

The PDF, $f_X : \mathbb{R} \to \mathbb{R}$, is the continuous analog of the PMF.

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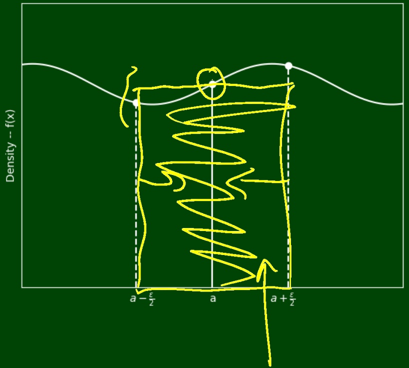
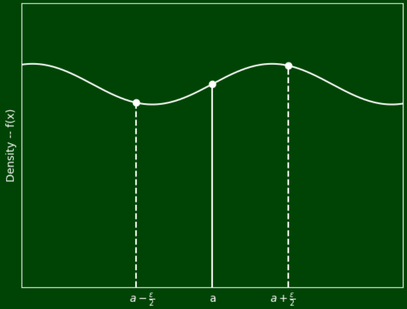
$$\Pr(a < X \leq b) = \int_a^b f_X(x)\, dx \text{ for } a \leq b$$



Density -- f(x)
$a-\frac{\epsilon}{2}$
a
$a+\frac{\epsilon}{2}$



$$
\begin{aligned}
\Pr\left(a-\frac{\varepsilon}{2} < X \le a+\frac{\varepsilon}{2}\right) &= \text{area under curve} \\
&= \int_{a-\frac{\varepsilon}{2}}^{a+\frac{\varepsilon}{2}} f_X(x)\,dx \\
&\approx \text{rectangle approximation} \\
&\approx f_X(a)\cdot\left(\left(a+\frac{\varepsilon}{2}\right)-\left(a-\frac{\varepsilon}{2}\right)\right) \\
&\approx f_X(a)\cdot\varepsilon
\end{aligned}
$$

Definition (Cumulative Distribution Function (CDF))

The CDF, $F_X : \mathbb{R} \to \mathbb{R}$, is the cumulative distribution to a point starting at $-\infty$. That is:

$$F_X(a) = \Pr(X \le a) = \int_{-\infty}^{a} f_x(x)\,dx$$

$$\Pr(a < X \le b) = \int_a^b f_x(x)\,dx$$

$$= F_x(b) - F_x(a)$$

$$f_x(x) = \frac{d}{dx} F_x(x)$$

CDF

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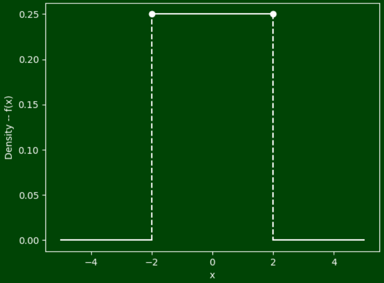
There is a relationship between f_X and F_X:

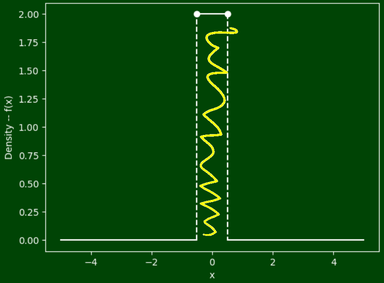
$$f_X(x) = \frac{d}{dx} F_X(x)$$

Uniform PDF

Uniform$(-2, 2)$ PDF



Uniform$(-1/2, 1/2)$ PDF



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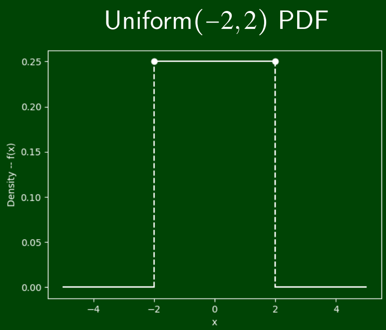
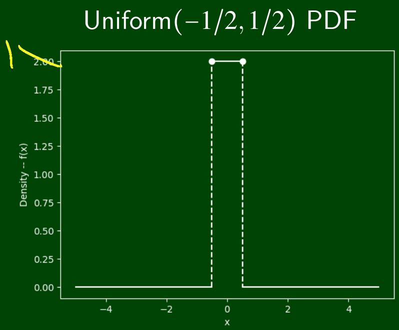
not a valid PDF.

$1 \cdot 2 \neq 1$





For $a < b$, $X \sim \text{Uniform}(a,b)$ iff

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

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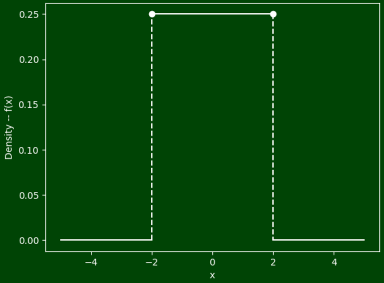

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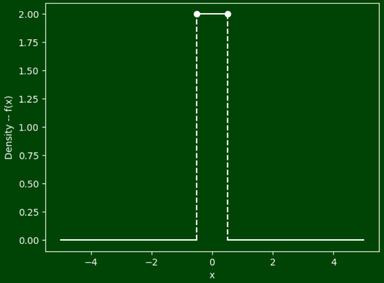

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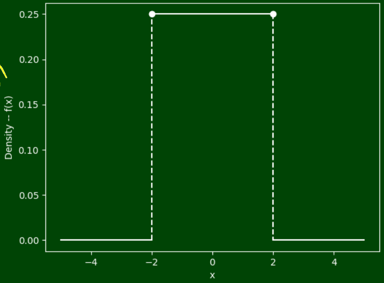

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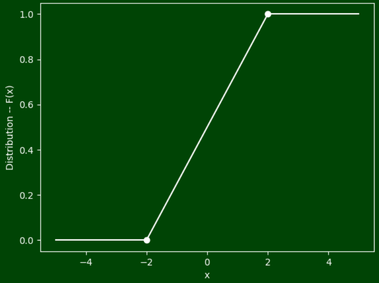

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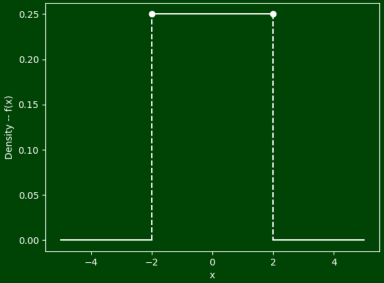
Recall: $f_X(x) = \frac{d}{dx} F_X(x)$

$$F_X(z) = \int_{\cancel{-\infty}\ a}^{z} f_x(x)\,dx = \frac{z-a}{b-a} \quad \text{if } a \le z \le b$$

Uniform CDF

Uniform$(-2,2)$ PDF



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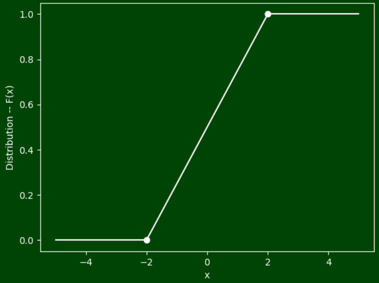

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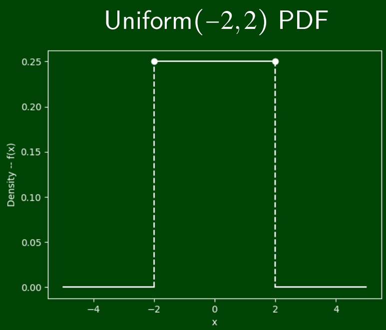

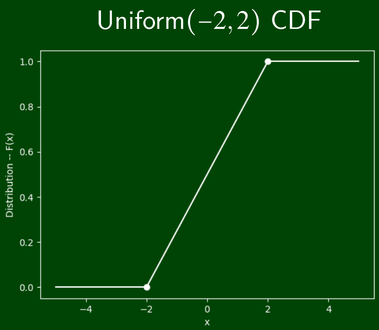

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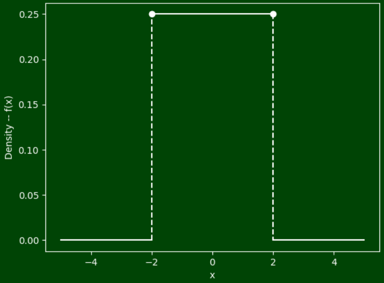

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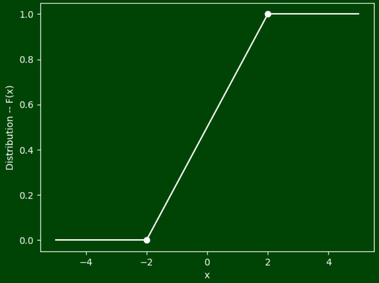

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$$F_X(z) = \int_{-\infty}^{z} f_X(x)\,dx = \int_a^z \frac{1}{b-a}\,dx = \frac{1}{b-a}\left(x\Big|_a^z\right) = \frac{z-a}{b-a}$$

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$$\Pr(x \le X \le y) = F_X(y) - F_X(x) = \frac{y-a}{b-a} - \frac{x-a}{b-a} = \frac{y-x}{b-a}$$

OR

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OR

$$\begin{aligned}
\Pr(x \le X \le y) &= \int_x^y f_X(x)\,dx \\
&= \int_x^y \frac{1}{b-a}\,dx \\
&= \frac{1}{b-a}\int_x^y dx \\
&= \frac{1}{b-a}\left(x\big|_x^y\right) \\
&= \frac{y-x}{b-a}
\end{aligned}$$

Uniform: $\mathbb{E}[X]$

$$\mathbb{E}[X] = \sum_{x=0}^{\infty} x \, p_X(x)$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx = \left. \frac{x^2}{2(b-a)} \right|_a^b$$

$$f_X(x) = \frac{1}{b-a}$$

Uniform: $\mathbb{E}[X]$

$$
\begin{aligned}
\mathbb{E}[X] &= \int_{-\infty}^{\infty} x f_X(x)\,dx \\
&= \int_a^b x f_X(x)\,dx \\
&= \int_a^b x \left(\frac{1}{b-a}\right) dx \\
&= \frac{1}{b-a} \int_a^b x\,dx \\
&= \frac{1}{b-a} \cdot \frac{1}{2} x^2 \Big|_a^b \\
&= \frac{1}{b-a} \left(\frac{1}{2} b^2 - \frac{1}{2} a^2\right) \\
&= \frac{1}{b-a} \left(\frac{(b-a)(b+a)}{2}\right) \\
&= \frac{b+a}{2}
\end{aligned}
$$

Uniform: $\mathbb{E}[X^2]$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$\mathbb{E}[X^2] = \int_a^b \frac{x^2}{b-a}\,dx = x^3\frac{1}{3}\left(\frac{1}{b-a}\right)\Big|_a^b$$

$$=$$

Uniform: $\mathbb{E}[X^2]$

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \\
&= \int_a^b x^2 f_X(x)\, dx \\
&= \int_a^b x^2 \left(\frac{1}{b-a}\right) dx \\
&= \frac{1}{b-a} \int_a^b x^2\, dx \\
&= \frac{1}{b-a} \cdot \frac{1}{3} x^3 \Big|_a^b \\
&= \frac{1}{b-a} \left(\frac{1}{3} b^3 - \frac{1}{3} a^3\right) \\
&= \frac{1}{b-a} \left(\frac{(b-a)(a^2+ab+b^2)}{3}\right) \\
&= \frac{a^2+ab+b^2}{3}
\end{aligned}$$

Uniform: Var(X)

$$\mathbb{E}[X] = \frac{b+a}{2}$$

$$\mathbb{E}\left[X^2\right] = \frac{a^2+ab+b^2}{3}$$

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$$\begin{aligned}
\text{Var}(X) &= \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2 \\
&= \frac{4a^2+4ab+4b^2-3(b+a)^2}{12} \\
&= \frac{4a^2+4ab+4b^2-3b^2-6ab-3a^2}{12} \\
&= \frac{a^2-2ab+b^2}{12} \\
&= \frac{(b-a)^2}{12}
\end{aligned}$$