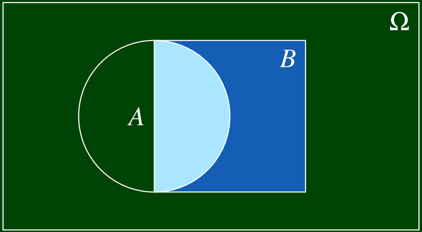
CSE 312

Foundations of Computing II

Conditional Probability & Law of Total Probability



Consider the following experiment:

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die1 = RollDie(6)
die2 = RollDie(6)
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If we already know $\texttt{die1} = 1$, what is $\Pr(\texttt{die1} + \texttt{die2} = 2)$?

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$$\Pr(\texttt{die1} + \texttt{die2} = 2) = \Pr(\texttt{die2} = 1) = \frac{1}{6}$$

If we already know $\texttt{die1} + \texttt{die2} = 3$, what is $\Pr(\texttt{die1} = 1) = \Pr(E)$?

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$$\begin{array}{llllllll}
\Omega = \{ & (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), & \\
 & (2,1), & (2,2) & (2,3), & (2,4), & (2,5), & (2,6), & \\
 & (3,1), & (3,2) & (3,3), & (3,4), & (3,5), & (3,6), & \\
 & (4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6), & \\
 & (5,1), & (5,2), & (5,3) & (5,4), & (5,5), & (5,6), & \\
 & (6,1), & (6,2), & (6,3) & (6,4), & (6,5), & (6,6) & \}
\end{array}$$

$$\begin{array}{llllllll}
E = \{ & (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) & \}
\end{array}$$

$\Pr(\texttt{die1} = 1 \text{ if } \texttt{die1} + \texttt{die2} = 3) =$

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$$E = \{ \quad (1,1), \quad (1,2), \quad (1,3), \quad (1,4), \quad (1,5), \quad (1,6) \quad \}$$

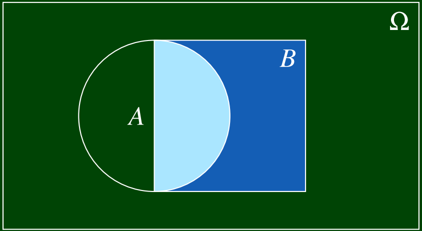
$\Pr(\texttt{die1} = 1 \text{ if } \texttt{die1} + \texttt{die2} = 3) = \frac{1}{2}$

Conditional Probability Definition

Definition (Conditional Probability)

We say the $\Pr(A \mid B)$ is "the probability of A given B", and we can calculate it as:

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$



$\Pr(A \mid B)$ "restricts" the sample space to B and asks for $\Pr(A)$ **with that information**.

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```

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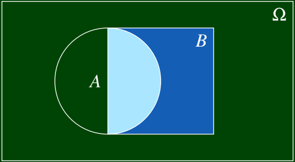

Way 1: $\Pr(\texttt{die1} = 1 \mid \texttt{die1} + \texttt{die2} = 3) = \dfrac{|A \cap B|}{|B|} =$

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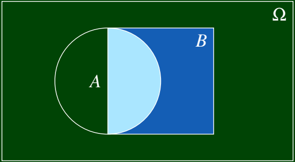

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Conditional Probability Example with Formulas

```
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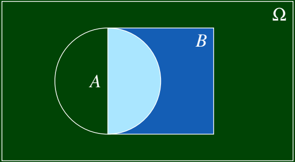

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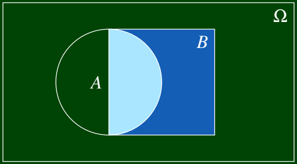
Another Example

Way 1



Way 2

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

```
roll = RollDie(6)
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What is $\Pr(\texttt{roll} = 5 \mid \texttt{roll} \text{ is odd})$?

- $A = \{5\}$
- $B = \{1,3,5\}$

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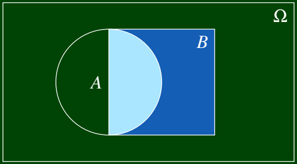

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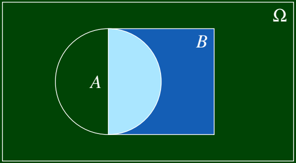

Way 2

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Bit string with m 1's and n 0's sent on the network with all arrangements of bits equally likely.
E = first bit received is a 0
F = k of the first r bits received are 0's

What is $\Pr(E \mid F)$?

- E = set of $m+n$ tuples with the first bit 0
- F = set of $m+n$ tuples with k 0's in the first r bits

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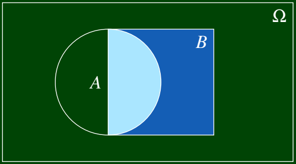

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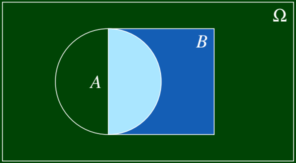

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- $\Pr(E \mid F) = \dfrac{|E \cap F|}{|F|} = \dfrac{\binom{r-1}{k-1}\binom{n+m-r}{n-k}}{\binom{r}{k}\binom{n+m-r}{n-k}}$

We have three bags with coins: one has two gold coins, one has two silver coins, and the third has one gold coin and one silver coin.

We randomly choose one of the bags, and we randomly select a coin from the bag. It turns out to be gold.
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- $\Pr(G_1 \cap G_2) = 1/3$ (the coins must be in the gold-gold bag)
- $\Pr(G_2 \mid G_1) = \frac{\Pr(G_1 \cap G_2)}{\Pr(G_1)} = \frac{2}{3}$

Definition (Product Rule)

$$\Pr(A_1 \cap A_2 \cap A_3) = \Pr(A_1)\Pr(A_2 \mid A_1)\Pr(A_3 \mid A_1 \cap A_2)$$

We can derive this by applying $\cancel{\Pr(X \cap Y) = \Pr(X)\Pr(X \mid Y)}$ twice:

$\Pr(A \cap (B \cap C)) =$

$$\Pr(Y \mid X) = \frac{\Pr(Y \cap X)}{\Pr(X)}$$

Y given X

$$\Pr(X \cap Y) = \Pr(X)\Pr(Y \mid X)$$

$$\Pr(A \cap (B \cap C)) =$$

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Example

Three cards are drawn from an ordinary 52-card deck without replacement. What is the probability that none of them is a heart?

Let A_i be the event that the ith card is not a heart.

We're looking for $\Pr(A_1 \cap A_2 \cap A_3)$.

$$\Pr(A_1 \cap A_2 \cap A_3) = \frac{\binom{39}{3}}{\binom{52}{3}}$$

$$\Pr(A_1 \cap A_2 \cap A_3) = \Pr(A_1)\Pr(A_2 \mid A_1)\Pr(A_3 \mid A_1 \cap A_2)$$

$$= \frac{39}{52}\,\frac{38}{51}\,\frac{37}{50}$$

Law of Total Probability

Definition (Law of Total Probability)

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid \overline{B})\Pr(\overline{B})$$

Note that $E \cap F$ and $E \cap \overline{F}$ are disjoint. So:

$$\begin{aligned} \Pr\big((E \cap F) \cup (E \cap \overline{F})\big) &= \Pr(E \cap F) + \Pr(E \cap \overline{F}) \\ \Pr(E) &= \Pr(E \cap F) + \Pr(E \cap \overline{F}) \\ &= \Pr(A \mid B)\Pr(B) + \Pr(A \mid \overline{B})\Pr(\overline{B}) \end{aligned}$$

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\Pr(E) &= \Pr(E \cap F) + \Pr\left(E \cap \overline{F}\right) \\
&= \Pr(A \mid B)\Pr(B) + \Pr\left(A \mid \overline{B}\right)\Pr\left(\overline{B}\right)
\end{aligned}$$

Definition (General Law of Total Probability)

$$\Pr(A) = \sum_{i=0}^{n} \Pr(A \mid B_i)\Pr(B_i)$$

Consider the following experiment:

```
coin = FlipCoin(1/2)
if coin == HEADS:
   die = RollDie(6)
else:
   die = RollDie(3)
```

What is $\Pr(\mathtt{die} = 1)$?

$A = (\text{die} = 1)$

$B = (\text{coin is HEADS})$

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid \bar{B})\Pr(\bar{B})$$

$$\frac{1}{6} \quad \frac{1}{2} \qquad \frac{1}{3} \quad \frac{1}{2}$$

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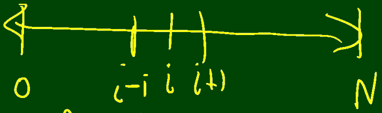
What is $\Pr(\texttt{die} = 1)$?

$$\begin{aligned}\Pr(\texttt{die} = 1) &= \Pr(\texttt{die} = 1 \mid \texttt{coin} = \texttt{HEADS}) * \Pr(\texttt{coin} = \texttt{HEADS}) \\ &\quad + \Pr(\texttt{die} = 1 \mid \texttt{coin} = \texttt{TAILS}) * \Pr(\texttt{coin} = \texttt{TAILS}) \\ &= \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{2}\right)\end{aligned}$$

A has i dollars; B has $N-i$
Flip a coin. If HEADS, A wins \$1 from B. If TAILS, B wins \$1 from A.
Repeat until A or B has all N dollars.

```
A = i
B = N - i
while A < N and B < N:
   result = FlipCoin(1/2)
   A += result == HEADS ? 1 : -1;
   B += result == HEADS ? -1 : 1;
```

Let E_i be the event "A wins starting with \$$i$". What is $\Pr(E_i)$?



Let B be A wins coin flip (HEADS)

$\Pr(E_i) = \Pr(E_{i+1} \mid H)\Pr(H) + \Pr(E_{i-1} \mid \bar{H})\Pr(\bar{H})$

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Let E_i be the event "A wins starting with \$$i$". What is $\Pr(E_i)$?

$$\begin{aligned}\Pr(E_i) &= \Pr(E_i \mid H)\Pr(H) + \Pr(E_i \mid T)\Pr(T) \\ &= \frac{1}{2}\left(\Pr(E_i \mid H) + \Pr(E_i \mid T)\right) \\ &= \frac{1}{2}\left(\Pr(E_{i+1}) + \Pr(E_{i-1})\right)\end{aligned}$$

So, $\Pr(E_{i+1}) = 2\Pr(E_i) - \Pr(E_{i-1})$.

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$$
\begin{aligned}
p_{i+1} &= 2p_i - p_{i-1} \\
p_2 &= 2p_1 - p_0 = 2p_1 \\
p_3 &= 2p_2 - p_1 = 2 \times 2p_1 - 1 \times p_1 = 3p_1 \\
&\dots \\
p_i &= 2p_{i-1} - p_{i-2} = 2(i-1)p_1 - (i-2)p_1 = \underline{ip_1}
\end{aligned}
$$

```
while FlipCoin(1/2) ≠ HEADS:
    print "Hello!"
```

p_i = prob. terminate on i flips

$$\Pr(\text{halt}) = \boxed{\Pr(\text{halt} \mid H)}\,\underbrace{\Pr(H)}_{\frac{1}{2}} + \boxed{\Pr(\text{halt} \mid T)}\,\underbrace{\Pr(T)}_{\frac{1}{2}}$$

$$= \frac{1}{2}\left(1 + \Pr(\text{halts})\right)$$

A Recursive Example

```
while FlipCoin(1/2) ≠ HEADS:
    print "Hello!"
```

$$\begin{aligned}\Pr(\texttt{halt}) &= \Pr(\texttt{halt} \mid \texttt{flip } i \texttt{ = HEADS})\Pr(\texttt{flip } i \texttt{ = HEADS}) \\ &\quad + \Pr(\texttt{halt} \mid \texttt{flip } i \texttt{ = TAILS})\Pr(\texttt{flip } i \texttt{ = TAILS}) \\ &= \Pr(\texttt{halt})\left(\frac{1}{2}\right) + 1\left(\frac{1}{2}\right)\end{aligned}$$

$$\frac{\Pr(\texttt{halt})}{2} = \frac{1}{2}$$

$$\Pr(\texttt{halt}) = 1$$