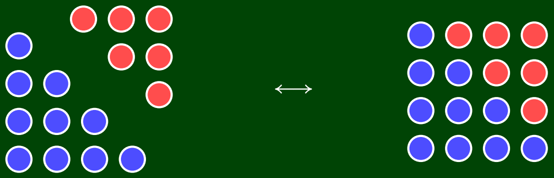
CSE 312

Foundations of Computing II

Counting in Two Ways



- Extra practice section in EEB 037 from 3:30pm - 4:30pm on Thursday. You don't need to sign up; just show up. Yes, the problems will be posted on the website.
- HW Grading
- Adam's Open Door Policy
- HW 1 Out

Outline

1. What is Counting in Two Ways?
2. Examples
3. Binomial Theorem

A New Proof Strategy: Counting in Two Ways

Definition (Counting in Two Ways)

If we have a set X and natural numbers n, m, then if $n = |X|$ and $m = |X|$, then $n = |X| = m$.

Okay, duh, but...

Definition (Triangle Numbers)

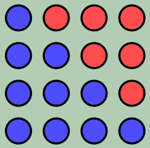
The nth Triangle Number, $\triangle_n = 1 + 2 + \cdots + n$.

Let's prove that $n^2 = \triangle_n + \triangle_{n-1}$.

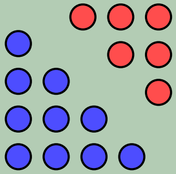
Proof 1: Induction

We did enough of that in CSE 311.

Proof 2: Counting in Two Ways



$\longleftrightarrow$



Make a square with n dots on each side.

Make a square by combining a triangle of height n and a triangle of height $n-1$.

Outline

Prove that $\binom{n}{k} = \binom{n}{n-k}$.

Set

We claim that both sides of this identity count the number of **committees of size k we can make out of n possible members**.

Way 1: Use the definition

Note that the left side counts this by definition.

Way 2: Be Clever!

The right side chooses $n-k$ **people to be excluded from the committee**. This leaves behind k to be included.

How many ways are there to arrange n people in a row?

Way 1: Use $n!$

Just order them: there are $n!$ ways to do this.

Way 2: Use $\binom{n}{k}$

Choose the first k people in the row. Then, order them. Then, order the remaining people.

There are $\binom{n}{k}$ ways to do the first step, $k!$ ways to do the second step, and $(n-k)!$ ways to do the third step. By the rule of product, there are $\binom{n}{k}k!(n-k)!$ ways to order n people.

The Point...

We've just shown that $n! = \binom{n}{k}k!(n-k)!$; that is: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

The first question to answer is what set both sides are counting.

- If one of the sides is $\binom{n}{k}$, we should think...
  Committees of size k from n people.
- If one of the sides is 312^k, we should think...
  Strings of length k with 312 possibilities for each character.
- If one of the sides is $n!$, we should think...
  Arrangments of n distinct items.

How many ways are there to arrange n people in a circle?

We can pull the same trick. Temporarily denote this number as C_n. Let's answer the same question as before, but this time using C_n:

How many ways are there to arrange n doggies in a row?

Way 1: Use $n!$

Just order them: there are $n!$ ways to do this.

Way 2: Use C_n

First, place the doggies in a circle. (There are C_n ways to do this)
Then, split the circle open by choosing a single doggie to lead the line. (There are n ways to do this)

The Point...

We've just shown that $n! = C_n n$; that is: $C_n = \frac{n!}{n} = (n-1)!$

And Another. . .

Prove

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

by counting in two ways.

Both Sides Count. . .

The number of ways to choose a group of k doggies from n doggies.

Way 1: Use $\binom{n}{k}$

Just choose them: there are $\binom{n}{k}$ ways to do this.

Way 2: Get Clever!

Partition the choices into two sets: (1) groups without **Hopper**, (2) groups with **Hopper**.

There are $\binom{n-1}{k}$ ways to choose a group that DOES NOT have **Hopper**.

There are $\binom{n-1}{k-1}$ ways to choose a group that DOES have **Hopper**.

How many ways are there to re-arrange the letters in the word APPLE? Call this number N.

Both Sides Count...

The number of ways to rearrange $\text{AP}_1\text{P}_2\text{LE}$.

Way 1: Use $n!$

Just arrange them: there are $n!$ ways to do this.

Way 2: Get Clever!

Arrange the letters of APPLE (there are N ways to do this). Arrange the order of the P's (there are $2!$ ways to do this).

So, $n! = N \times 2!$. So, $N = \frac{n!}{2!}$

Outline

1. What is Counting in Two Ways?
2. Examples
3. Binomial Theorem

Binomial Theorem?

Definition (Binomial Theorem)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

A Specific Case

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Let's prove this combinatorially!

A Specific Case

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Proof Sketch.

Set: Binary strings of length n.
Left: By definition.
Right: Partition on how many bits are 0. $\square$

Even Subsets

$$\sum_{k\geq 0}\binom{n}{2k} = 2^{n-1}$$

Proof Sketch.

Set: Even subsets of $[n]$.
Left: Partition on how many elements are in the set.
Right: Decide whether i is in the set or not up to $n-1$–then, at that point the set we chose is either even or odd. If it's even, we can't include the last element; if it's odd, we must include the last element. □

The Real Deal

Definition (Binomial Theorem)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Proof Sketch.

Set: Consider x odd numbers and y even numbers. We line up n numbers (with replacement!) in a row.
Left: For each spot in the row, choose any number.
Right: Partition on how many odd numbers are in the row (k). There are n spots these k numbers could go; choose k of them. Then, determine which k odd numbers are in the row and which $n-k$ even numbers are in the row. □