random variables

let X = index of

$X : \Omega \to \mathbb{R}$

$Pr(X=k) = Pr(\{\omega \mid X(\omega)=k\})$

A *random variable* is a *numeric function* of the outcome of an experiment, not the outcome itself.

Ex.

Let *H* be the *number* of Heads when 20 coins are tossed

Let *T* be the *total* of 2 dice rolls

Let *X* be the *number* of coin tosses needed to see 1st head

Note: even if the underlying experiment has "equally likely outcomes," the associated random variable *may not*

Outcome	*X = #H*	*P(X)*
TT	0	P(X=0) = 1/4
TH	1	P(X=1) = 1/2
HT	1	
HH	2	P(X=2) = 1/4

$\Omega =$ collection all subsets of size 3.

$\Pr(\{5, 9, 13\}) = \frac{1}{\binom{20}{3}}$

$|\Omega| = \binom{20}{3}$

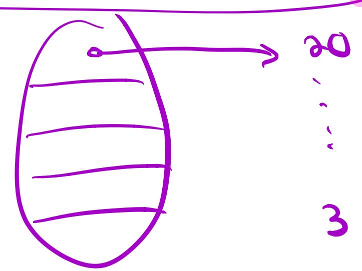
$\Pr(\omega) =$

20 balls numbered 1, 2, ..., 20

Draw a subset of size 3 at random.

Let X = the maximum of the numbers on those 3 balls

What is $\boxed{P(X \geq 17)}$



$$\Pr(X=20) = \frac{\binom{19}{2}}{\binom{20}{3}}$$

$$\Pr(X=18) = \frac{\binom{17}{2}}{\binom{20}{3}}$$

20 balls numbered 1, 2, ..., 20

Draw a subset of size 3 at random.

Let X = the maximum of the numbers on those 3 balls

What is P(X ≥ 17)

$$
\begin{aligned}
P(X=20) &= \tbinom{19}{2} \Big/ \tbinom{20}{3} = \frac{3}{20} = 0.150 \\
P(X=19) &= \tbinom{18}{2} \Big/ \tbinom{20}{3} = \frac{18 \cdot 17/2!}{20 \cdot 19 \cdot 18/3!} \approx 0.134 \\
&\ \ \vdots
\end{aligned}
$$

$$\textstyle\sum_{i=17}^{20} P(X=i) \approx 0.508$$

$$P(X \geq 17) = 1 - P(X < 17) = 1 - \binom{16}{3} \Big/ \binom{20}{3} \approx 0.508$$

Flip a (biased) coin (probability p of Heads) repeatedly until 1st head observed

What is the sample space?

$\Omega = \{ \underset{1}{H}, \underset{2}{TH}, \underset{3}{TTH}, \ldots \}$

$X(\omega)$

How many flips? Let X be that number.

$P(X=1) = p$

$P(X=2) = (1-p) \cdot p$

$P(X=3) = (1-p)^2 p$.

…

$P(X=i) = (1-p)^{i-1} p$.

Flip a (biased) coin repeatedly until 1st head observed

How many flips? Let X be that number.

- $P(X=1) = P(H) = p$
- $P(X=2) = P(TH) = (1-p)p$
- $P(X=3) = P(TTH) = (1-p)^2p$
- ...
- $P(X=i) = P(T^{i-1}H) = (1-p)^{i-1}p$

$$\sum_{i \geq 0} x^i = \frac{1}{1-x},$$

when $|x| < 1$

memorize me!

A function whose domain is the sample space.

A *discrete* random variable is one taking on a *countable* number of possible values.

Ex:

X = sum of 3 dice, $3 \leq X \leq 18$, $X \in N$

Y = position of 1st head in seq of coin flips, $1 \leq Y$, $Y \in N$

Z = largest prime factor of $(1+Y)$, $Z \in \{2, 3, 5, 7, 11, ...\}$

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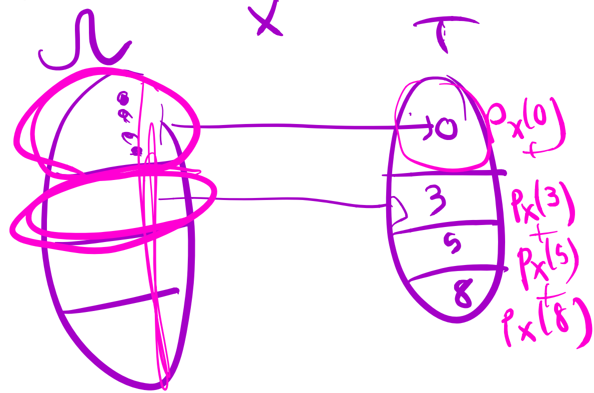


Definition: If X is a discrete random variable taking on values from a countable set $T \subseteq \mathcal{R}$, then

$$p_X(a) = \begin{cases} P(X = a) & \text{for } a \in T \\ 0 & \text{otherwise} \end{cases}$$

is called the *probability mass function*. Note: $\sum_{a \in T} p_X(a) = 1$

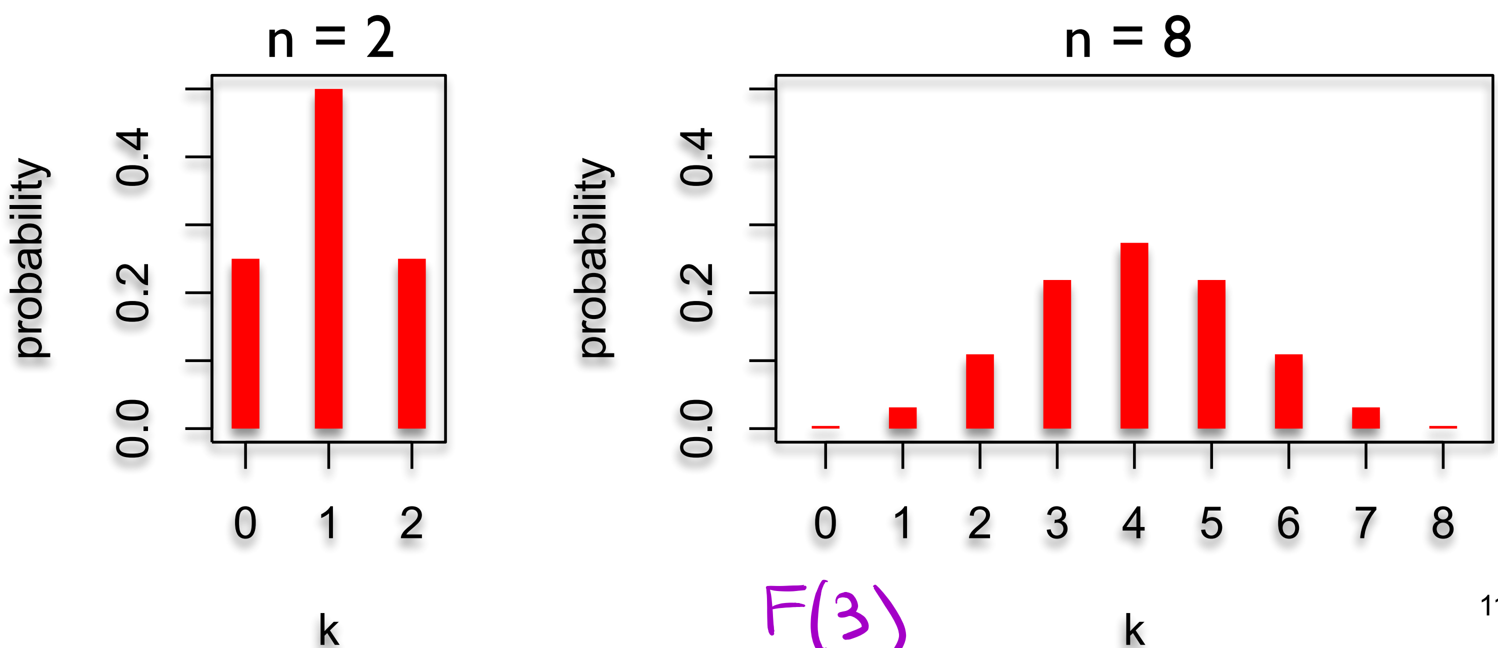
Let X be the number of heads in n independent coin tosses, each with probability p of heads.

$$\boxed{p_X(k)} = Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Let X be the number of heads observed in n coin flips

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \text{ where } p = P(H)$$

Probability mass function (p = ½):



F(3)

$= Pr(X \le 3)$

The *cumulative distribution function* for a random variable X is the function F: $\mathcal{R} \rightarrow [0,1]$ defined by

1 2 3

$$\boxed{F(a) = P[X \le a]}$$

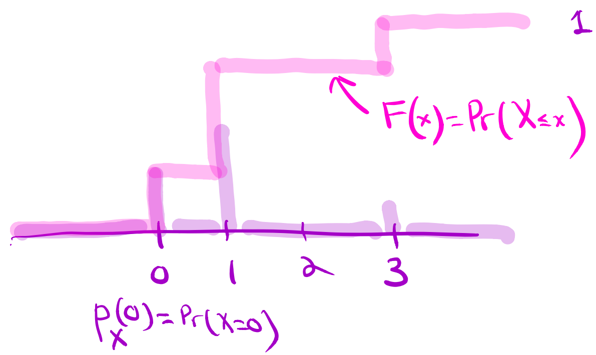
Ex: 3 students; homework returned according to random permutation.

X is number of homeworks returned to their correct homework.

What is probability mass function?

Cumulative distribution function?

Prob	outcome	X
1/6	123	3
1/6	132	1
1/6	213	1
1/6	231	0
1/6	312	0
1/6	321	1



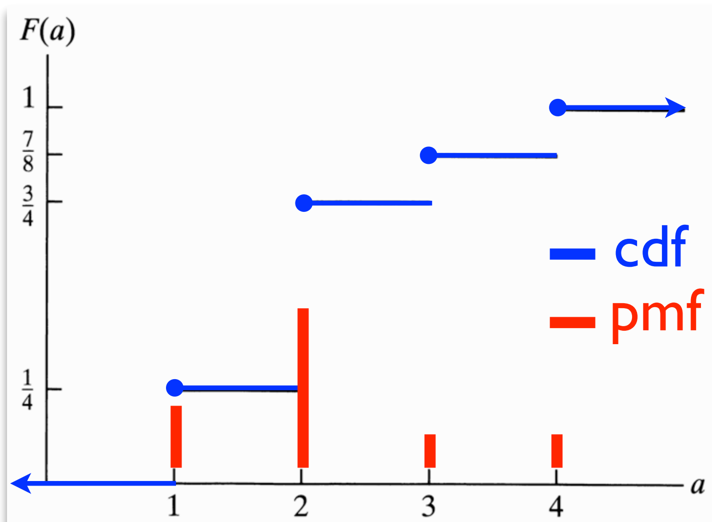
The *cumulative distribution function* for a random variable X is the function $F: \mathcal{R} \to [0,1]$ defined by

$$\boxed{F(a) = P[X \le a]}$$

Ex: if X has probability mass function given by:

$$p(1) = \frac{1}{4} \qquad p(2) = \frac{1}{2} \qquad p(3) = \frac{1}{8} \qquad p(4) = \frac{1}{8}$$

$$F(a) = \begin{cases} 0 & a < 1 \\ \frac{1}{4} & 1 \le a < 2 \\ \frac{3}{4} & 2 \le a < 3 \\ \frac{7}{8} & 3 \le a < 4 \\ 1 & 4 \le a \end{cases}$$



NB: for discrete random variables, be careful about "≤" vs "<"

expectation

For a discrete r.v. X with p.m.f. $p(\bullet)$, the *expectation of X*, aka *expected value* or *mean*, is

$$\boxed{E[X] = \Sigma_x\, x p(x)}$$

average of random values, *weighted* by their respective probabilities

For the equally-likely outcomes case, this is just the average of the possible random values of X

For *un*equally-likely outcomes, it is again the average of the possible random values of X, weighted by their respective probabilities

Ex 1: Let X = value seen rolling a fair die $p(1), p(2), ..., p(6) = 1/6$

$$E(X) = \sum_{i=1}^{6} i \underbrace{Pr(X=i)}_{p(i)} = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + \cdots + 6\cdot\frac{1}{6} = 3.5$$

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Ex 1: Let X = value seen rolling a fair die $p(1), p(2), ..., p(6) = 1/6$

$$E[X] = \sum_{i=1}^{6} ip(i) = \frac{1}{6}(1+2+\cdots+6) = \frac{21}{6} = 3.5$$

Ex 2: Coin flip; X = +1 if H (win \$1), -1 if T (lose \$1)

$$E(X) = 1 \cdot \underbrace{Pr(X=1)}_{\frac{1}{2}} + (-1)\underbrace{Pr(X=-1)}_{\frac{1}{2}} = 0$$

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Ex 1: Let X = value seen rolling a fair die $p(1), p(2), ..., p(6) = 1/6$

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Ex 2: Coin flip; X = +1 if H (win \$1), -1 if T (lose \$1)

$$E[X] = (+1) \cdot p(+1) + (-1) \cdot p(-1) = 1 \cdot (1/2) + (-1) \cdot (1/2) = 0$$

For a discrete r.v. X with p.m.f. $p(\bullet)$, the *expectation of X*, aka *expected value* or *mean,* is

$$E[X] = \Sigma_x\, x p(x)$$

average of random values, weighted by their respective probabilities

Another view: A 2-person gambling game. If X is how much you win playing the game once, how much would you expect to win, on average, per game, when repeatedly playing?

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Ex 1: Let X = value seen rolling a fair die $p(1), p(2), ..., p(6) = 1/6$

If you win X dollars for that roll, how much do you expect to win?

$$E[X] = \sum_{i=1}^{6} i p(i) = \frac{1}{6}(1 + 2 + \cdots + 6) = \frac{21}{6} = 3.5$$

Ex 2: Coin flip; X = +1 if H (win \$1), -1 if T (lose \$1)

$E[X] = (+1)\bullet p(+1) + (-1)\bullet p(-1) = 1\bullet(1/2) + (-1)\bullet(1/2) = 0$

"a fair game": in repeated play you expect to win as much as you lose. Long term net gain/loss = 0.

Let X be the number of flips up to & including 1st head observed in repeated flips of a biased coin (with probability p of coming up heads). If I pay you \$1 per flip, how much money would you expect to make?

X: # flips till see a H's.

$$E(X) = \sum_{i=1}^{\infty} i \, \overbrace{Pr(X=i)}^{p_X(i)}$$

$$= \sum_{i=1}^{\infty} i \, (1-p)^{i-1} p = p \underbrace{\sum_{i=1}^{\infty} i \, (\underbrace{1-p}_{x})^{i-1}} = p \, \frac{1}{(1-(1-p))^2} = \frac{1}{p}$$

$$\sum_{i=1}^{\infty} i x^{i-1} = \frac{1}{(1-x)^2}$$

$$\sum_{i \geq 0} x^i = \frac{1}{1-x},$$

when $|x| < 1$

memorize me!

Let X be the number of flips up to & including 1st head observed in repeated flips of a biased coin. If I pay you \$1 per flip, how much money would you expect to make?

$$\begin{aligned} P(H) &= p; \quad P(T) = 1-p = q \\ p(i) &= pq^{i-1} \qquad \leftarrow \textit{PMF} \\ E[X] &= \textstyle\sum_{i\geq 1} ip(i) = \sum_{i\geq 1} ipq^{i-1} = p\sum_{i\geq 1} iq^{i-1} \qquad (*) \end{aligned}$$

A calculus trick:

$$\sum_{i\geq 1} iy^{i-1} = \sum_{i\geq 1} \frac{d}{dy} y^i = \sum_{i\geq 0} \frac{d}{dy} y^i = \frac{d}{dy} \sum_{i\geq 0} y^i = \frac{d}{dy} \frac{1}{1-y} = \frac{1}{(1-y)^2}$$

$dy^0/dy = 0$

So $(*)$ becomes:

$$E[X] = p\sum_{i\geq 1} iq^{i-1} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

E.g.:

p=1/2; on average head every 2nd flip

p=1/10; on average, head every 10th flip.

How much would you pay to play?

Let X be the number of heads observed in n repeated flips of a biased coin. If I pay you \$1 per head, how much money would you expect to make?

$$P_X(i)$$

$$E(X) = \sum_{i=0}^{n} i \Pr(X=i)$$

$$= \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i}$$

$n = 100$

E.g.:

p=1/2 50

p=1/10 10

Let X be the number of heads observed in n repeated flips of a biased coin. If I pay you $1 per head, how much money would you expect to make?

$$
\begin{aligned}
E[X] &= \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i} \\
&= \sum_{i=1}^{n} i \binom{n}{i} p^i (1-p)^{n-i} \\
&= \sum_{i=1}^{n} n \binom{n-1}{i-1} p^i (1-p)^{n-i} \\
&= np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i} \\
&= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \\
&= np(p + (1-p))^{n-1} = np
\end{aligned}
$$

E.g.:

p=1/2; on average, n/2 heads

p=1/10; on average, n/10 heads

For a discrete r.v. X with p.m.f. $p(\bullet)$, the *expectation of X*, aka *expected value* or *mean*, is

$$E[X] = \Sigma_x \, x p(x) = \sum_{\substack{x \\ \text{values } X \text{ takes}}} x \Pr(X=x)$$

Another view:

$$\cancel{E[X] = \sum_{s \in S} X(s) \cdot p(s)}$$

$$E(X) = \sum_{\omega \in \Omega} X(\omega) \Pr(\omega)$$