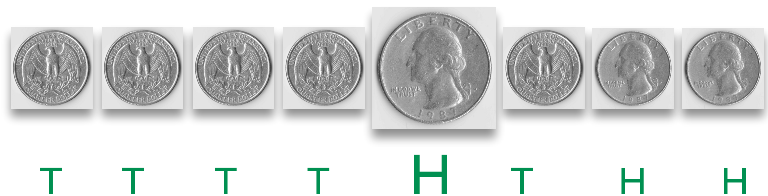
random variables



let X = index of

Ω, $Pr(\cdot)$ $X: \Omega \rightarrow \mathbb{R}$

A *random variable* is a *numeric function* of the outcome of an experiment, not the outcome itself.

Ex.

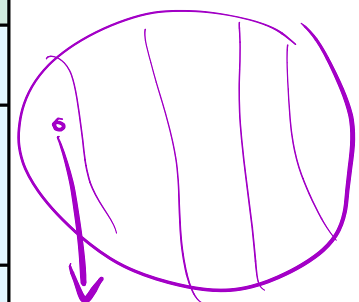
Let *H* be the *number* of Heads when 20 coins are tossed

Let *T* be the *total* of 2 dice rolls

Let *X* be the *number* of coin tosses needed to see 1st head

Note: even if the underlying experiment has "equally likely outcomes," the associated random variable *may not*

Outcome	*X = #H*	*P(X)*
TT	0	P(X=0) = 1/4
TH	1	P(X=1) = 1/2
HT	1	
HH	2	P(X=2) = 1/4



$Pr(X=a) = Pr(\{\omega \mid X(\omega)=a\})$

20 balls numbered 1, 2, ..., 20

Draw a subset of size 3 at random.

Let X = the maximum of the numbers on those 3 balls

What is $P(X \geq 17)$

20 balls numbered 1, 2, ..., 20

Draw a subset of size 3 at random.

Let X = the maximum of the numbers on those 3 balls

What is P(X ≥ 17)

$$P(X=20) \quad = \quad \binom{19}{2} \Big/ \binom{20}{3} \quad = \quad \frac{3}{20} \quad = \quad 0.150$$

$$P(X=19) \quad = \quad \binom{18}{2} \Big/ \binom{20}{3} \quad = \quad \frac{18 \cdot 17/2!}{20 \cdot 19 \cdot 18/3!} \quad \approx \quad 0.134$$

$$\vdots$$

$$\sum_{i=17}^{20} P(X=i) \;\approx\; 0.508$$

$$P(X \geq 17) \quad = \quad 1 - P(X < 17) \;=\; 1 - \binom{16}{3} \Big/ \binom{20}{3} \;\approx\; 0.508$$

Flip a (biased) coin (probability p of Heads) repeatedly until 1st head observed

What is the sample space?

How many flips? Let X be that number.

$P(X=1) =$

$P(X=2) =$

$P(X=3) =$

…

$P(X=i)$

Flip a (biased) coin repeatedly until 1st head observed

How many flips? Let X be that number.

$P(X=1) = P(H) = p$

$P(X=2) = P(TH) = (1-p)p$

$P(X=3) = P(TTH) = (1-p)^2p$

...

$P(X=i) = P(T^{i-1}H) = (1-p)^{i-1}p$

$$\boxed{\sum_{i \geq 0} x^i = \frac{1}{1-x}, \quad \text{when } |x| < 1}$$

memorize me!

A *discrete* random variable is one taking on a *countable* number of possible values.

Ex:

X = sum of 3 dice, $3 \leq X \leq 18$, $X \in N$

Y = position of 1st head in seq of coin flips, $1 \leq Y$, $Y \in N$

Z = largest prime factor of $(1+Y)$, $Z \in \{2, 3, 5, 7, 11, \ldots\}$

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Ex:

X = sum of 3 dice, $3 \leq X \leq 18$, $X \in N$

Definition: If X is a discrete random variable taking on values from a countable set $T \subseteq \mathcal{R}$, then

$$p_X(a) = \begin{cases} P(X = a) & \text{for } a \in T \\ 0 & \text{otherwise} \end{cases}$$

is called the *probability mass function*. Note: $\sum_{a \in T} p_X(a) = 1$

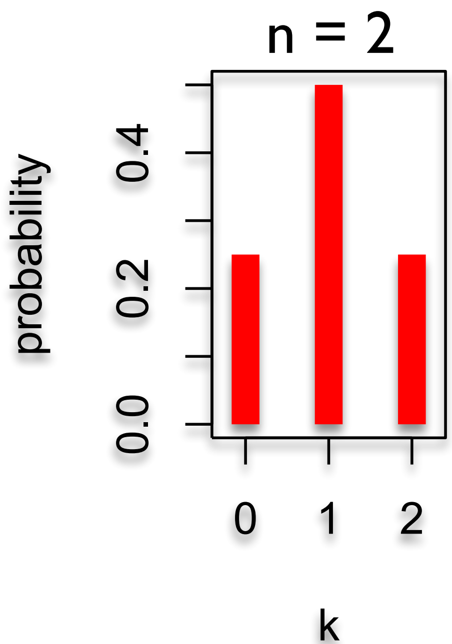
Let X be the number of heads in n independent coin tosses, each with probability p of heads.

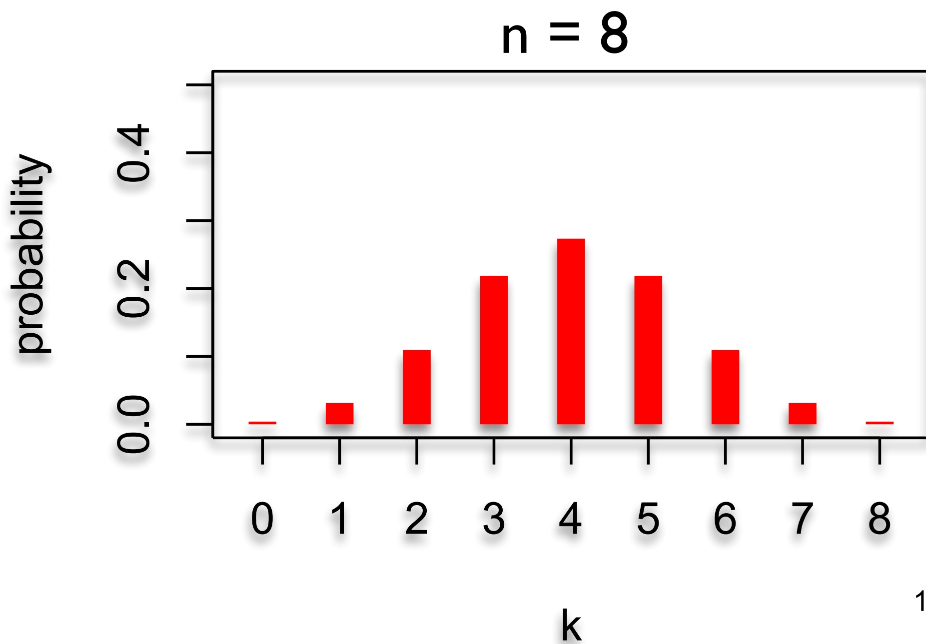
$$p_X(x) = Pr(X = k)$$

Let X be the number of heads observed in n coin flips

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \text{ where } p = P(H)$$

Probability mass function (p = ½):





The *cumulative distribution function* for a random variable X is the function F: $\mathcal{R} \rightarrow [0,1]$ defined by

$$\boxed{F(a) = P[X \leq a]}$$

Ex: 3 students; homework returned according to random permutation.

X is number of homeworks returned to their correct homework.

Prob	outcome	X
1/6	123	3
1/6	132	1
1/6	213	1
1/6	231	0
1/6	312	0
1/6	321	1

What is probability mass function?

Cumulative distribution function?

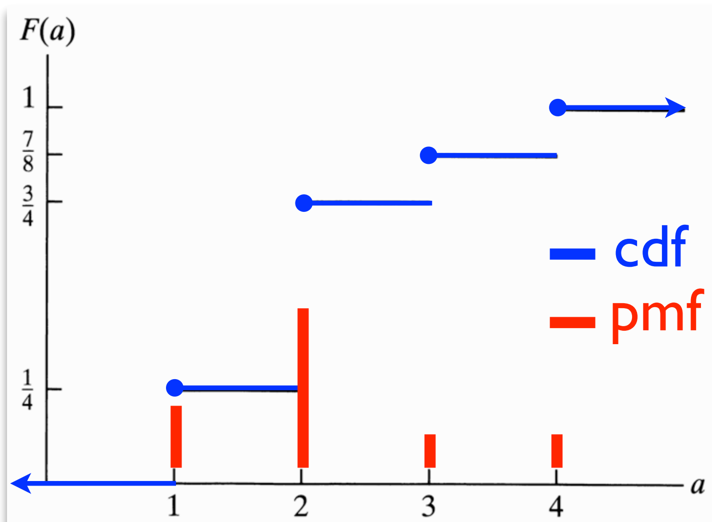
The *cumulative distribution function* for a random variable X is the function $F: \mathcal{R} \to [0,1]$ defined by

$$\boxed{F(a) = P[X \le a]}$$

Ex: if X has probability mass function given by:

$$p(1) = \tfrac{1}{4} \quad p(2) = \tfrac{1}{2} \quad p(3) = \tfrac{1}{8} \quad p(4) = \tfrac{1}{8}$$

$$F(a) = \begin{cases} 0 & a < 1 \\ \frac{1}{4} & 1 \le a < 2 \\ \frac{3}{4} & 2 \le a < 3 \\ \frac{7}{8} & 3 \le a < 4 \\ 1 & 4 \le a \end{cases}$$



NB: for discrete random variables, be careful about "≤" vs "<"

expectation

For a discrete r.v. X with p.m.f. p(•), the *expectation of X*, aka *expected value* or *mean*, is

$$\boxed{E[X] = \Sigma_x \, x p(x)}$$

average of random values, *weighted* by their respective probabilities

For the equally-likely outcomes case, this is just the average of the possible random values of X

For *un*equally-likely outcomes, it is again the average of the possible random values of X, weighted by their respective probabilities

Ex 1: Let X = value seen rolling a fair die $p(1), p(2), ..., p(6) = 1/6$

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Ex 1: Let X = value seen rolling a fair die $p(1), p(2), ..., p(6) = 1/6$

$$E[X] = \sum_{i=1}^{6} ip(i) = \frac{1}{6}(1 + 2 + \cdots + 6) = \frac{21}{6} = 3.5$$

Ex 2: Coin flip; X = +1 if H (win \$1), -1 if T (lose \$1)

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$$E[X] = \sum_{i=1}^{6} i p(i) = \frac{1}{6}(1+2+\cdots+6) = \frac{21}{6} = 3.5$$

Ex 2: Coin flip; X = +1 if H (win \$1), -1 if T (lose \$1)

$$E[X] = (+1)\bullet p(+1) + (-1)\bullet p(-1) = 1\bullet(1/2) + (-1)\bullet(1/2) = 0$$

For a discrete r.v. X with p.m.f. $p(\bullet)$, the *expectation of X*, aka *expected value* or *mean*, is

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average of random values, weighted by their respective probabilities

Another view: A 2-person gambling game. If X is how much you win playing the game once, how much would you expect to win, on average, per game, when repeatedly playing?

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Ex 1: Let X = value seen rolling a fair die $p(1), p(2), \ldots, p(6) = 1/6$

If you win X dollars for that roll, how much do you expect to win?

$$E[X] = \sum_{i=1}^{6} ip(i) = \frac{1}{6}(1 + 2 + \cdots + 6) = \frac{21}{6} = 3.5$$

Ex 2: Coin flip; X = +1 if H (win \$1), -1 if T (lose \$1)

$E[X] = (+1)\bullet p(+1) + (-1)\bullet p(-1) = 1\bullet(1/2) + (-1)\bullet(1/2) = 0$

"a fair game": in repeated play you expect to win as much as you lose. Long term net gain/loss = 0.

Let X be the number of flips up to & including 1st head observed in repeated flips of a biased coin (with probability p of coming up heads). If I pay you \$1 per flip, how much money would you expect to make?

$$\sum_{i \geq 0} x^i = \frac{1}{1-x},$$

$$\text{when } |x| < 1$$

memorize me!

Let X be the number of flips up to & including 1st head observed in repeated flips of a biased coin. If I pay you \$1 per flip, how much money would you expect to make?

$$\begin{aligned} P(H) &= p; \quad P(T) = 1 - p = q \\ p(i) &= pq^{i-1} \quad \leftarrow \textit{PMF} \\ E[X] &= \textstyle\sum_{i\geq 1} ip(i) = \sum_{i\geq 1} ipq^{i-1} = p\sum_{i\geq 1} iq^{i-1} \qquad (*) \end{aligned}$$

A calculus trick:

$$\sum_{i\geq 1} iy^{i-1} = \sum_{i\geq 1} \frac{d}{dy} y^i = \sum_{i\geq 0} \frac{d}{dy} y^i = \frac{d}{dy} \sum_{i\geq 0} y^i = \frac{d}{dy} \frac{1}{1-y} = \frac{1}{(1-y)^2}$$

$dy^0/dy = 0$

So $(*)$ becomes:

$$E[X] = p\sum_{i\geq 1} iq^{i-1} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

E.g.:

p=1/2; on average head every 2nd flip
p=1/10; on average, head every 10th flip.

How much would you pay to play?

Let X be the number of heads observed in n repeated flips of a biased coin. If I pay you $1 per head, how much money would you expect to make?

E.g.:

p=1/2

p=1/10

Let X be the number of heads observed in n repeated flips of a biased coin. If I pay you \$1 per head, how much money would you expect to make?

E.g.:

p=1/2; on average, n/2 heads

p=1/10; on average, n/10 heads

$$
\begin{aligned}
E[X] &= \sum_{i=0}^{n} i \binom{n}{i} p^i (1-p)^{n-i} \\
&= \sum_{i=1}^{n} i \binom{n}{i} p^i (1-p)^{n-i} \\
&= \sum_{i=1}^{n} n \binom{n-1}{i-1} p^i (1-p)^{n-i} \\
&= np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i} \\
&= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} \\
&= np(p + (1-p))^{n-1} = np
\end{aligned}
$$

For a discrete r.v. X with p.m.f. $p(\bullet)$, the *expectation of X*, aka *expected value* or *mean*, is

$$E[X] = \Sigma_x \, x p(x)$$

Another view:

$$E[X] = \sum_{s \in S} X(s) \cdot p(s)$$

expectation of a *function* of a random variable

Calculating $E[g(X)]$: $E[X] = \Sigma_x \, x p(x)$

$E[X] = \sum_{s \in S} X(s) \cdot p(s)$

$Y = g(X)$ is a new r.v. Calculate $P[Y=j]$, then apply defn:

X = number of people who get their homework back

$Y = g(X) = X^2 \bmod 2$

Prob	outcome	X
1/6	123	3
1/6	132	1
1/6	213	1
1/6	231	0
1/6	312	0
1/6	321	1

Way 1

Calculating $E[g(X)]$:

$Y=g(X)$ is a new r.v. Calculate $P[Y=j]$, then apply defn:

X = sum of 2 dice rolls

i	$p(i) = P[X=i]$	$i \cdot p(i)$
2	1/36	2/36
3	2/36	6/36
4	3/36	12/36
5	4/36	20/36
6	5/36	30/36
7	6/36	42/36
8	5/36	40/36
9	4/36	36/36
10	3/36	30/36
11	2/36	22/36
12	1/36	12/36

$E[X] = \Sigma_i \, ip(i) =$ 252/36 $= 7$

$Y = g(X) = X \bmod 5$

j	$q(j) = P[Y = j]$	$j \cdot q(j)$
0	4/36+3/36 = 7/36	0/36
1	5/36+2/36 = 7/36	7/36
2	1/36+6/36+1/36 = 8/36	16/36
3	2/36+5/36 = 7/36	21/36
4	3/36+4/36 = 7/36	28/36

$E[Y] = \Sigma_j \, jq(j) =$ 72/36 $= 2$

Way 2

expectation of a *function* of a random variable

Calculating $E[g(X)]$: Another way – *add in a different order, using* $P[X=...]$ *instead of calculating* $P[Y=...]$

X = sum of 2 dice rolls

i	$p(i) = P[X=i]$	$g(i)\bullet p(i)$
2	1/36	2/36
3	2/36	6/36
4	3/36	12/36
5	4/36	0/36
6	5/36	5/36
7	6/36	12/36
8	5/36	15/36
9	4/36	16/36
10	3/36	0/36
11	2/36	2/36
12	1/36	2/36

$E[g(X)] = \Sigma_i\, g(i)p(i) = \boxed{72/36} = 2$

$Y = g(X) = X \bmod 5$

j	$q(j) = P[Y = j]$	$j\bullet q(j)$
0	4/36+3/36 = 7/36	0/36
1	5/36+2/36 = 7/36	7/36
2	1/36+6/36+1/36 = 8/36	16/36
3	2/36+5/36 = 7/36	21/36
4	3/36+4/36 = 7/36	28/36

$E[Y] = \Sigma_j\, jq(j) = \boxed{72/36} = 2$

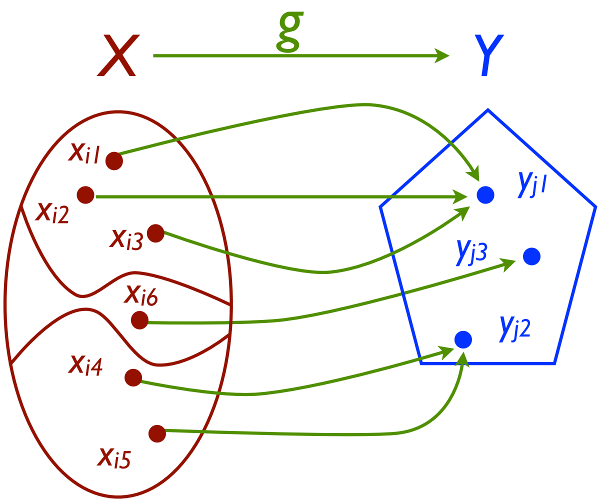
BT pg.84-85

Above example is not a fluke.

Theorem: if $Y = g(X)$, then $E[Y] = \Sigma_i\, g(x_i)p(x_i)$, where

x_i, i = 1, 2, ... are all possible values of X.

Proof: Let y_j, j = 1, 2, ... be all possible values of Y.



Note that $S_j = \{ x_i \mid g(x_i)=y_j \}$ is a *partition* of the domain of g.

BT pg.84-85

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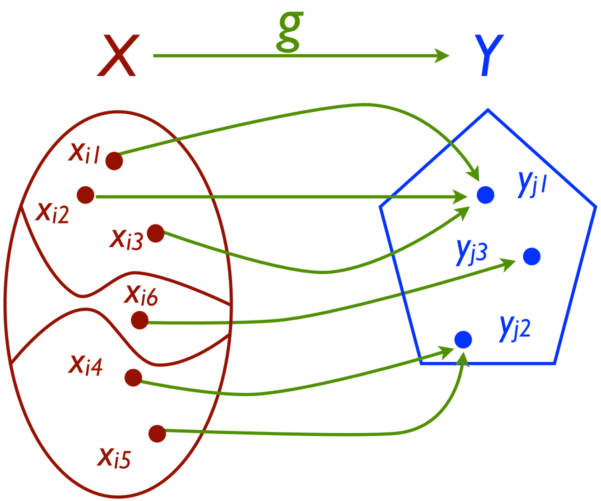

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$$
\begin{aligned}
\sum_i g(x_i)p(x_i) &= \sum_j \sum_{i:g(x_i)=y_j} g(x_i)p(x_i) \\
&= \sum_j \sum_{i:g(x_i)=y_j} y_j p(x_i) \\
&= \sum_j y_j \sum_{i:g(x_i)=y_j} p(x_i) \\
&= \sum_j y_j P\{g(X) = y_j\} \\
&= E[g(X)]
\end{aligned}
$$

A & *B* each bet \$1, then flip 2 coins:

HH	*A* wins \$2
HT	Each takes back \$1
TH	
TT	*B* wins \$2

Let X be *A*'s net gain: +1, 0, -1, resp.:

$P(X = +1) = 1/4$
$P(X = 0) = 1/2$
$P(X = -1) = 1/4$

What is $E[X]$?

$$E[X] = 1 \cdot 1/4 + 0 \cdot 1/2 + (-1) \cdot 1/4 = 0$$

What is $E[X^2]$?

$$E[X^2] = 1^2 \cdot 1/4 + 0^2 \cdot 1/2 + (-1)^2 \cdot 1/4 = 1/2$$

Big Deal Note:
$E[X^2] \neq E[X]^2$