random variables

let $X$ = index of
random variables

A random variable is a numeric function of the outcome of an experiment, not the outcome itself.

Ex.

Let $H$ be the number of Heads when 20 coins are tossed
Let $T$ be the total of 2 dice rolls
Let $X$ be the number of coin tosses needed to see 1\textsuperscript{st} head

Note: even if the underlying experiment has “equally likely outcomes,” the associated random variable may not

<table>
<thead>
<tr>
<th>Outcome</th>
<th>$X = #H$</th>
<th>$P(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>TT</td>
<td>0</td>
<td>$P(X=0) = 1/4$</td>
</tr>
<tr>
<td>TH</td>
<td>1</td>
<td>${P(X=1) = 1/2}$</td>
</tr>
<tr>
<td>HT</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>HH</td>
<td>2</td>
<td>$P(X=2) = 1/4$</td>
</tr>
</tbody>
</table>

$\Pr(X=a) = \Pr(\{\omega \mid X(\omega) = a\})$
20 balls numbered 1, 2, ..., 20
Draw a subset of size 3 at random.
Let $X = \text{the maximum of the numbers on those 3 balls}$
What is $P(X \geq 17)$
20 balls numbered 1, 2, ..., 20
Draw a subset of size 3 at random.
Let $X =$ the maximum of the numbers on those 3 balls
What is $P(X \geq 17)$

\[ P(X = 20) = \frac{\binom{19}{2}}{\binom{20}{3}} = \frac{3}{20} = 0.150 \]
\[ P(X = 19) = \frac{\binom{18}{2}}{\binom{20}{3}} = \frac{18\cdot17/2!}{20\cdot19\cdot18/3!} \approx 0.134 \]
\[ \vdots \]
\[ \sum_{i=17}^{20} P(X = i) \approx 0.508 \]

\[ P(X \geq 17) = 1 - P(X < 17) = 1 - \frac{\binom{16}{3}}{\binom{20}{3}} \approx 0.508 \]
Flip a (biased) coin (probability $p$ of Heads) repeatedly until 1st head observed

What is the sample space?

How many flips? Let $X$ be that number.

$P(X=1) =$

$P(X=2) =$

$P(X=3) =$

$\ldots$

$P(X=i)$
Flip a (biased) coin repeatedly until 1st head observed

How many flips? Let \( X \) be that number.

\[
P(X=1) = P(H) = p
\]

\[
P(X=2) = P(TH) = (1-p)p
\]

\[
P(X=3) = P(TTH) = (1-p)^2p
\]

..., 

\[
P(X=i) = P(T^{i-1}H) = (1-p)^{i-1}p
\]
A **discrete** random variable is one taking on a **countable** number of possible values.

Ex:

- \( X = \text{sum of 3 dice}, \ 3 \leq X \leq 18, \ X \in \mathbb{N} \)
- \( Y = \text{position of 1st head in seq of coin flips}, \ 1 \leq Y, \ Y \in \mathbb{N} \)
- \( Z = \text{largest prime factor of (1+Y)}, \ Z \in \{2, 3, 5, 7, 11, \ldots\} \)
A discrete random variable is one taking on a countable number of possible values.

Ex:

\[ X = \text{sum of 3 dice}, \quad 3 \leq X \leq 18, \quad X \in \mathbb{N} \]

**Definition:** If \( X \) is a discrete random variable taking on values from a countable set \( T \subseteq \mathbb{R} \), then

\[
p_X(a) = \begin{cases} 
  P(X = a) & \text{for } a \in T \\
  0 & \text{otherwise}
\end{cases}
\]

is called the **probability mass function**. Note: \( \sum_{a \in T} p_X(a) = 1 \)
Let $X$ be the number of heads in $n$ independent coin tosses, each with probability $p$ of heads.

$$p_X(x) = Pr(X = k)$$
Let $X$ be the number of heads observed in $n$ coin flips

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \text{ where } p = P(H)$$

Probability mass function ($p = \frac{1}{2}$):
The **cumulative distribution function** for a random variable $X$ is the function $F: \mathbb{R} \rightarrow [0,1]$ defined by

$$F(a) = P[X \leq a]$$

Ex: 3 students; homework returned according to random permutation.

$X$ is number of homeworks returned to their correct homework.

What is probability mass function? Cumulative distribution function?
The *cumulative distribution function* for a random variable $X$ is the function $F: \mathbb{R} \to [0,1]$ defined by

$$F(a) = P[X \leq a]$$

Ex: if $X$ has *probability mass function* given by:

$$p(1) = \frac{1}{4} \quad p(2) = \frac{1}{2} \quad p(3) = \frac{1}{8} \quad p(4) = \frac{1}{8}$$

$$F(a) = \begin{cases} 
0 & a < 1 \\
\frac{1}{4} & 1 \leq a < 2 \\
\frac{3}{4} & 2 \leq a < 3 \\
\frac{7}{8} & 3 \leq a < 4 \\
1 & 4 \leq a \end{cases}$$

NB: for discrete random variables, be careful about “$\leq$” vs “$<$”
expectation
For a discrete r.v. $X$ with p.m.f. $p(\cdot)$, the \textit{expectation of $X$}, aka \textit{expected value} or \textit{mean}, is

$$E[X] = \sum_x xp(x)$$

average of random values, \textit{weighted} by their respective probabilities

For the equally-likely outcomes case, this is just the average of the possible random values of $X$

For \textit{unequally-likely} outcomes, it is again the average of the possible random values of $X$, \textit{weighted by their respective probabilities}

Ex 1: Let $X =$ value seen rolling a fair die  $p(1), p(2), \ldots, p(6) = 1/6$
For a discrete r.v. $X$ with p.m.f. $p(\cdot)$, the expectation of $X$, aka expected value or mean, is

$$E[X] = \sum_x xp(x)$$

For the equally-likely outcomes case, this is just the average of the possible random values of $X$.

For unequally-likely outcomes, it is again the average of the possible random values of $X$, weighted by their respective probabilities.

Ex 1: Let $X =$ value seen rolling a fair die $p(1), p(2), \ldots, p(6) = 1/6$

$$E[X] = \sum_{i=1}^{6} ip(i) = \frac{1}{6} (1 + 2 + \cdots + 6) = \frac{21}{6} = 3.5$$

Ex 2: Coin flip; $X =$ +1 if H (win $\$1), -1 if T (lose $\$1)$
For a discrete r.v. $X$ with p.m.f. $p(\cdot)$, the \textit{expectation of $X$}, aka \textit{expected value} or \textit{mean}, is

$$E[X] = \sum_x xp(x)$$

\textit{average of random values, weighted by their respective probabilities}

\textbf{Ex 2:} Coin flip; $X = +1$ if H (win $\$1$), -1 if T (lose $\$1$)
For a discrete r.v. $X$ with p.m.f. $p(\cdot)$, the *expectation of $X$, aka expected value or mean*, is

$$E[X] = \sum_x x p(x)$$

average of random values, weighted by their respective probabilities

For the equally-likely outcomes case, this is just the average of the possible random values of $X$

For *unequally*-likely outcomes, it is again the average of the possible random values of $X$, *weighted by their respective probabilities*

**Ex 1:** Let $X = \text{value seen rolling a fair die}$ $p(1), p(2), \ldots, p(6) = 1/6$

$$E[X] = \sum_{i=1}^{6} ip(i) = \frac{1}{6}(1 + 2 + \cdots + 6) = \frac{21}{6} = 3.5$$

**Ex 2:** Coin flip; $X = +1$ if H (win $\$1), $-1$ if T (lose $\$1)$

$$E[X] = (+1)\cdot p(+1) + (-1)\cdot p(-1) = 1\cdot(1/2) +(-1)\cdot(1/2) = 0$$
For a discrete r.v. $X$ with p.m.f. $p(\cdot)$, the \textit{expectation of $X$}, aka \textit{expected value} or \textit{mean}, is

$$E[X] = \sum_x x p(x)$$

\textit{Another view:} A 2-person gambling game. If $X$ is how much you win playing the game once, how much would you expect to win, on average, per game, when repeatedly playing?
For a discrete r.v. \( X \) with p.m.f. \( p(\bullet) \), the \textit{expectation of} \( X \), aka \textit{expected value} or \textit{mean}, is

\[
E[X] = \sum_x x p(x)
\]

**Another view:** A 2-person gambling game. If \( X \) is how much you win playing the game once, how much would you expect to win, on average, per game, when repeatedly playing?

\textbf{Ex 1:} Let \( X = \text{value seen rolling a fair die} \ p(1), p(2), \ldots, p(6) = 1/6 \)

If you win \( X \) dollars for that roll, how much do you expect to win?

\[
E[X] = \sum_{i=1}^{6} ip(i) = \frac{1}{6} (1 + 2 + \cdots + 6) = \frac{21}{6} = 3.5
\]

\textbf{Ex 2:} Coin flip; \( X = +1 \) if H (win $1), -1 if T (lose $1)

\[
E[X] = (+1)\cdot p(+1) + (-1)\cdot p(-1) = 1\cdot(1/2) +(-1)\cdot(1/2) = 0
\]

“a fair game”: in repeated play you expect to win as much as you lose. Long term net gain/loss = 0.
Let $X$ be the number of flips up to & including 1st head observed in repeated flips of a biased coin (with probability $p$ of coming up heads). If I pay you $1 per flip, how much money would you expect to make?

$$
\sum_{i \geq 0} x^i = \frac{1}{1 - x},
$$

when $|x| < 1$

memorize me!
Let $X$ be the number of flips up to & including 1st head observed in repeated flips of a biased coin. If I pay you $1 per flip, how much money would you expect to make?

$$P(H) = p; \quad P(T) = 1 - p = q$$
$$p(i) = pq^{i-1} \leftarrow \text{PMF}$$
$$E[X] = \sum_{i \geq 1} ip(i) = \sum_{i \geq 1} ipq^{i-1} = p \sum_{i \geq 1} iq^{i-1} \quad (\ast)$$

A calculus trick:

$$\sum_{i \geq 1} iy^{i-1} = \sum_{i \geq 1} \frac{d}{dy} y^i = \sum_{i \geq 0} \frac{d}{dy} y^i = \frac{d}{dy} \sum_{i \geq 0} y^i = \frac{d}{dy} \frac{1}{1 - y} = \frac{1}{(1 - y)^2}$$

So (\ast) becomes:

$$E[X] = p \sum_{i \geq 1} iq^{i-1} = \frac{p}{(1 - q)^2} = \frac{p}{p^2} = \frac{1}{p} \quad \text{How much would you pay to play?}$$

E.g.:

$p = 1/2$; on average head every 2nd flip

$p = 1/10$; on average, head every 10th flip.
Let $X$ be the number of heads observed in $n$ repeated flips of a biased coin. If I pay you $1$ per head, how much money would you expect to make?

E.g.:

$p=1/2$

$p=1/10$
Let $X$ be the number of heads observed in $n$ repeated flips of a biased coin. If I pay you $1$ per head, how much money would you expect to make?

E.g.: $p=1/2$; on average, $n/2$ heads

$p=1/10$; on average, $n/10$ heads

$$E[X] = \sum_{i=0}^{n} \binom{n}{i} p^i (1 - p)^{n-i}$$

$$= \sum_{i=1}^{n} \binom{n}{i} p^i (1 - p)^{n-i}$$

$$= \sum_{i=1}^{n} n \binom{n-1}{i-1} p^i (1 - p)^{n-i}$$

$$= np \sum_{i=1}^{n} \binom{n-1}{i-1} p^{i-1} (1 - p)^{n-i}$$

$$= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^{j} (1 - p)^{n-1-j}$$

$$= np(p + (1 - p))^{n-1} = np$$
For a discrete r.v. $X$ with p.m.f. $p(\bullet)$, the *expectation of $X$*, aka *expected value* or *mean*, is

$$E[X] = \sum_x xp(x)$$

Another view:

$$E[X] = \sum_{s \in S} X(s) \cdot p(s)$$
Calculating $E[g(X)]$:

\[
E[X] = \sum_x x p(x) \\
E[X] = \sum_{s \in S} X(s) \cdot p(s)
\]

$Y=g(X)$ is a new r.v. Calculate $P[Y=j]$, then apply defn:

$X =$ number of people who get their homework back

$Y = g(X) = X^2 \mod 2$

<table>
<thead>
<tr>
<th>Prob</th>
<th>outcome</th>
<th>$X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/6</td>
<td>123</td>
<td>3</td>
</tr>
<tr>
<td>1/6</td>
<td>132</td>
<td>1</td>
</tr>
<tr>
<td>1/6</td>
<td>213</td>
<td>1</td>
</tr>
<tr>
<td>1/6</td>
<td>231</td>
<td>0</td>
</tr>
<tr>
<td>1/6</td>
<td>312</td>
<td>0</td>
</tr>
<tr>
<td>1/6</td>
<td>321</td>
<td>1</td>
</tr>
</tbody>
</table>
Calculating $E[g(X)]$:

$Y = g(X)$ is a new r.v. Calculate $P[Y = j]$, then apply defn:

$X = \text{sum of 2 dice rolls}$

$Y = g(X) = X \mod 5$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$p(i) = P[X=i]$</th>
<th>$i*p(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$1/36$</td>
<td>$2/36$</td>
</tr>
<tr>
<td>3</td>
<td>$2/36$</td>
<td>$6/36$</td>
</tr>
<tr>
<td>4</td>
<td>$3/36$</td>
<td>$12/36$</td>
</tr>
<tr>
<td>5</td>
<td>$4/36$</td>
<td>$20/36$</td>
</tr>
<tr>
<td>6</td>
<td>$5/36$</td>
<td>$30/36$</td>
</tr>
<tr>
<td>7</td>
<td>$6/36$</td>
<td>$42/36$</td>
</tr>
<tr>
<td>8</td>
<td>$5/36$</td>
<td>$40/36$</td>
</tr>
<tr>
<td>9</td>
<td>$4/36$</td>
<td>$36/36$</td>
</tr>
<tr>
<td>10</td>
<td>$3/36$</td>
<td>$30/36$</td>
</tr>
<tr>
<td>11</td>
<td>$2/36$</td>
<td>$22/36$</td>
</tr>
<tr>
<td>12</td>
<td>$1/36$</td>
<td>$12/36$</td>
</tr>
</tbody>
</table>

$E[X] = \sum_i i p(i) = 252/36 = 7$

<table>
<thead>
<tr>
<th>$j$</th>
<th>$q(j) = P[Y = j]$</th>
<th>$j*q(j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$4/36 + 3/36 = 7/36$</td>
<td>$0/36$</td>
</tr>
<tr>
<td>1</td>
<td>$5/36 + 2/36 = 7/36$</td>
<td>$7/36$</td>
</tr>
<tr>
<td>2</td>
<td>$1/36 + 6/36 + 1/36 = 8/36$</td>
<td>$16/36$</td>
</tr>
<tr>
<td>3</td>
<td>$2/36 + 5/36 = 7/36$</td>
<td>$21/36$</td>
</tr>
<tr>
<td>4</td>
<td>$3/36 + 4/36 = 7/36$</td>
<td>$28/36$</td>
</tr>
</tbody>
</table>

$E[Y] = \sum_j j q(j) = 72/36 = 2$
Calculating $E[g(X)]$: Another way – *add in a different order*, using $P[X=...]$ instead of calculating $P[Y=...]$

$X = \text{sum of 2 dice rolls}$

$Y = g(X) = \text{X mod 5}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$p(i) = P[X=i]$</th>
<th>$g(i)\cdot p(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0/36</td>
<td>0/36</td>
</tr>
<tr>
<td>1</td>
<td>1/36</td>
<td>1/36</td>
</tr>
<tr>
<td>2</td>
<td>2/36</td>
<td>6/36</td>
</tr>
<tr>
<td>3</td>
<td>3/36</td>
<td>12/36</td>
</tr>
<tr>
<td>4</td>
<td>4/36</td>
<td>0/36</td>
</tr>
<tr>
<td>5</td>
<td>5/36</td>
<td>5/36</td>
</tr>
<tr>
<td>6</td>
<td>6/36</td>
<td>12/36</td>
</tr>
<tr>
<td>7</td>
<td>7/36</td>
<td>15/36</td>
</tr>
<tr>
<td>8</td>
<td>8/36</td>
<td>16/36</td>
</tr>
<tr>
<td>9</td>
<td>9/36</td>
<td>0/36</td>
</tr>
<tr>
<td>10</td>
<td>1/36</td>
<td>2/36</td>
</tr>
<tr>
<td>11</td>
<td>2/36</td>
<td>2/36</td>
</tr>
<tr>
<td>12</td>
<td>1/36</td>
<td>2/36</td>
</tr>
</tbody>
</table>

$E[g(X)] = \sum_i g(i)p(i) = \frac{72}{36} = 2$

$E[Y] = \sum_j jq(j) = \frac{72}{36} = 2$
Above example is not a fluke.

**Theorem:** if $Y = g(X)$, then $E[Y] = \sum_i g(x_i)p(x_i)$, where $x_i, i = 1, 2, ...$ are all possible values of $X$.

**Proof:** Let $y_j, j = 1, 2, ...$ be all possible values of $Y$.

Note that $S_j = \{ x_i | g(x_i) = y_j \}$ is a *partition* of the domain of $g$. 

**Diagram:**

- $X$ is mapped to $Y$ through $g$.
- $x_{i1}, x_{i2}, x_{i3}, x_{i4}, x_{i5}, x_{i6}$ are the image sets of $X$.
- $y_{j1}, y_{j2}, y_{j3}$ are the image sets of $Y$. 
- The diagram shows the mapping from $X$ to $Y$ through $g$. 

Note that $S_j = \{ x_i | g(x_i) = y_j \}$ is a partition of the domain of $g$. 

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**BT pg.84-85**
Above example is not a fluke.

Theorem: if $Y = g(X)$, then $E[Y] = \sum_i g(x_i)p(x_i)$, where $x_i, i = 1, 2, \ldots$ are all possible values of $X$.

Proof: Let $y_j, j = 1, 2, \ldots$ be all possible values of $Y$.

$\sum_i g(x_i)p(x_i) = \sum_j \sum_{i: g(x_i) = y_j} g(x_i)p(x_i)$

$= \sum_j \sum_{i: g(x_i) = y_j} y_jp(x_i)$

$= \sum_j y_j \sum_{i: g(x_i) = y_j} p(x_i)$

$= \sum_j y_j \sum_{j} p(x_i)$

$= \sum_j y_j P\{g(X) = y_j\}$

$= E[g(X)]$

Note that $S_j = \{ x_i \mid g(x_i) = y_j \}$ is a partition of the domain of $g$. 

### BT pg.84-85

### Slide 49
properties of expectation

A & B each bet $1, then flip 2 coins:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>HH</td>
<td>A wins $2</td>
</tr>
<tr>
<td>HT</td>
<td>Each takes back $1</td>
</tr>
<tr>
<td>TH</td>
<td></td>
</tr>
<tr>
<td>TT</td>
<td>B wins $2</td>
</tr>
</tbody>
</table>

Let \( X \) be A’s net gain: +1, 0, -1, resp.:

\[
E[X] = 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{4} = 0
\]

What is \( E[X^2] \)?

\[E[X^2] = 1^2 \cdot \frac{1}{4} + 0^2 \cdot \frac{1}{2} + (-1)^2 \cdot \frac{1}{4} = \frac{1}{2}\]

Big Deal Note:
\[E[X^2] \neq E[X]^2\]