Sample space: S = set of all potential outcomes of experiment

E.g., flip two coins: S = {(H,H), (H,T), (T,H), (T,T)}

Events: **E ⊆ S** is an arbitrary subset of the sample space

≥1 head in 2 flips: E = {(H,H), (H,T), (T,H)} S =

Probability:

A function from subsets of S to real numbers – Pr: $2^S \rightarrow [0,1]$

Probability Axioms:

Axiom 1 (Non-negativity): $0 \leq \Pr(E)$

Axiom 2 (Normalization): $\Pr(S) = 1$

Axiom 3 (Additivity): $EF = \emptyset \Rightarrow \Pr(E \cup F) = \Pr(E) + \Pr(F)$

Simplest case: sample spaces with equally likely outcomes.

Coin flips:	S = {Heads, Tails}
Flipping two coins:	S = {(H,H),(H,T),(T,H),(T,T)}
Roll of 6-sided die:	S = {1, 2, 3, 4, 5, 6}

Pr(each outcome) = $\dfrac{1}{|S|}$

uniform distribution

In that case,

$$\Pr(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{|E|}{|S|}$$

Ω = all 5 card hands

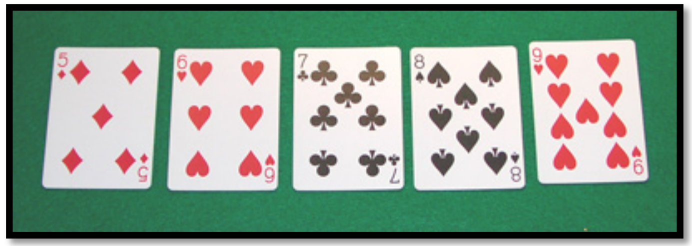
any straight in poker

$Pr(\omega) = \frac{1}{|\Omega|} = \frac{1}{\binom{52}{5}}$

Consider 5 card poker hands.

A "straight" is 5 consecutive rank cards of any suit

What is Pr(straight) ?



$|\Omega| = \binom{52}{5}$

$|E| = 10 \cdot \binom{4}{1}^5$

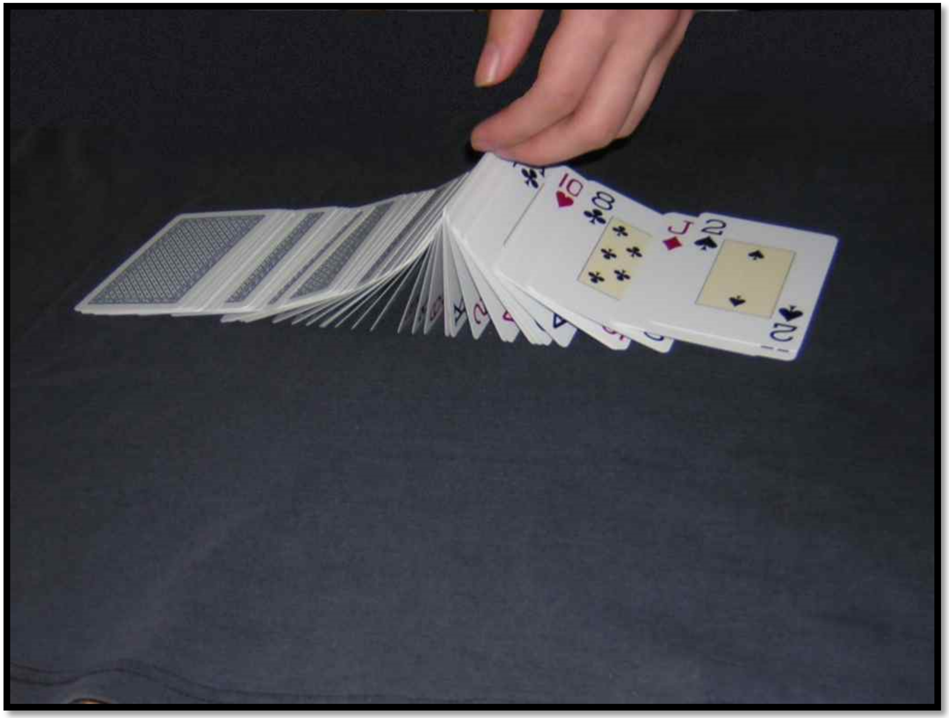
E: set of outcomes that are straights

$$\text{Pr(straight)} = \frac{10\binom{4}{1}^5}{\binom{52}{5}} \approx 0.00394$$

$Pr(E) = \frac{|E|}{|\Omega|}$

When you submit your homework on gradescope, you <u>must</u> assign page numbers to each problem

card flipping



card flipping

Ω : all possible permutations of 52 cards

$Pr(\omega) = \frac{1}{52!}$

52 card deck. Cards flipped one at a time.

After first ace (of any suit) appears, consider next card

Pr($\underbrace{\text{next card = ace of spades}}_{E_1}$) < Pr($\underbrace{\text{next card = 2 of clubs}}_{E_2}$) ?

E_1 ___ ___ ___ A A♠ ___ ___ ___

E_2 ___ ___ ___ A 2♣ ___ ___ ___

$$Pr(E_1) = \frac{|E_1|}{52!} = \frac{51!}{52!} = \frac{1}{52} = Pr(E_2)$$

$Pr(E_1) \; \substack{> \\ = \\ <} \; Pr(E_2)$?

A♥ A♠ $E_{1♥}$

A♦ A♠ $E_{1♦}$

A♣ A♠ $E_{1♣}$

$|E_{1♥}|$

___ ___ ___ A♥ A♠ ___

ith

$\binom{48}{i-1}(i-1)! \ldots$

card flipping

52 card deck. Cards flipped one at a time.

After first ace (of any suit) appears, consider next card

Pr(next card = ace of spades) < Pr(next card = 2 of clubs) ?

Case 1: Take Ace of Spades out of deck

Shuffle remaining 51 cards, add ace of spades after first ace

$|S| = 52!$ (all cards shuffled)

$|E| = 51!$ (only 1 place ace of spades can be added)

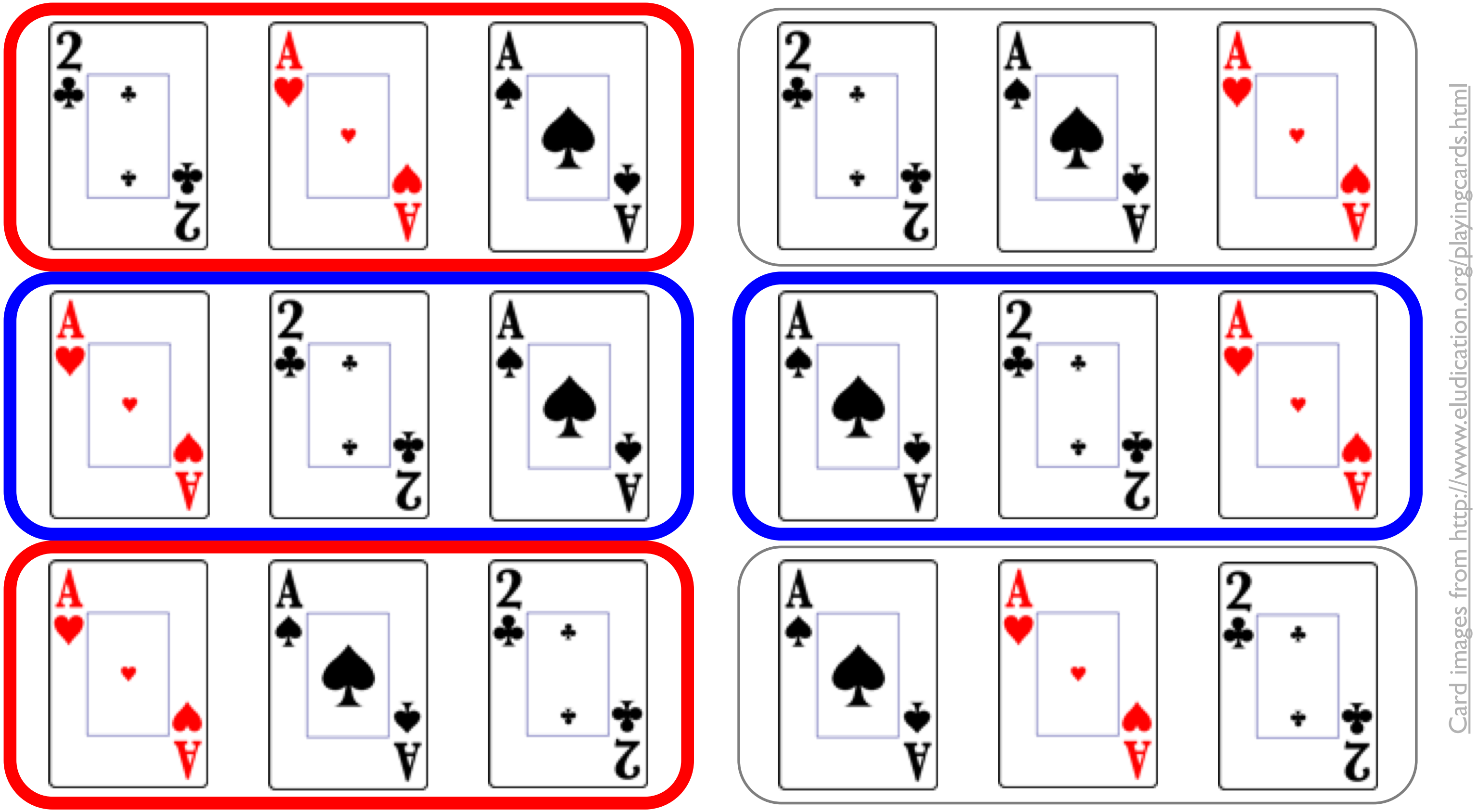
Case 2: Do the same thing with the 2 of clubs

$|S|$ and $|E|$ have same size

So,

Pr(next = Ace of spades) = Pr(next = 2 of clubs) = 1/52

Ace of Spades: 2/6 2 of Clubs: 2/6



Theory is the same for a 3-card deck; Pr = 2!/3! = 1/3

birthdays

What is the probability that, of n people, none share the same birthday?

Ω : set of possible birthdays for each of n people.

$|\Omega| = 365^n$

E: no 2 people share same bday

$A_1 \quad A_2, \quad --- \quad A_n$

$Pr(E) = \frac{|E|}{|\Omega|} = \frac{365 \cdot 364 \cdots (365-n+1)}{365^n}$

$A_1 \quad A_2$

March 19 March 20

March 20 March 19

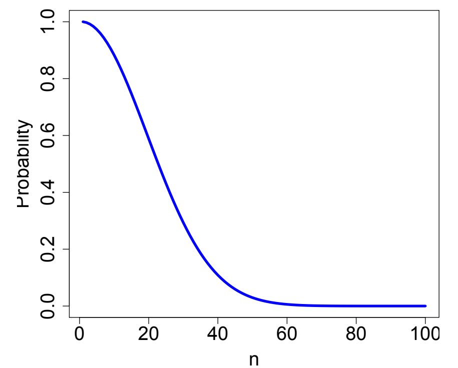
What is the probability that, of n people, none share the same birthday?

$|S| = (365)^n$

$|E| = (365)(364)(363)\cdots(365\text{-}n+1)$

Pr(no matching birthdays) = |E|/|S|

$= (365)(364)\ldots(365\text{-}n+1)/(365)^n$



Some values of n…

n = 23: Pr(no matching birthdays) < 0.5

n = 77: Pr(no matching birthdays) < 1/5000

n = 100: Pr(no matching birthdays) < 1/3,000,000

n = 150: Pr(…) < 1/3,000,000,000,000,000

n = 366?

Pr = 0

Above formula gives this, since

$$(365)(364)\ldots(365-n+1)/(365)^n == 0$$

when n = 366 (or greater).

Even easier to see via pigeon hole principle.

What is the probability that, of n people, none share the same birthday as you? me

March 19

$|\Omega| = 365^n$

$\Pr(\underbrace{\exists \text{ no person w/ birthday March 19}}_{E}) = \frac{|E|}{|\Omega|} = \frac{364^n}{365^n}$

What is the probability that, of n people, none share the same birthday as you?

$|S| = (365)^n$

$|E| = (364)^n$

Pr(no birthdays matches yours) = $|E|/|S|$

$= (364)^n/(365)^n$

Some values of n…

n = 23: Pr(no matching birthdays) $\approx$ 0.9388

n = 77: Pr(no matching birthdays) $\approx$ 0.8096

n = 253: Pr(no matching birthdays) $\approx$ 0.4995

Probability that a random 7 digit numbers (decimal) has at least $\underbrace{\text{one repeating digit?}}_{\exists \text{ some digit that appears twice or more.}}$ (allowed to have leading zeros).

0, 1, …, 9

$$|\Omega| = 10^7$$

E: at least one repeating digit

$$\Pr(E) = \frac{|E|}{10^7} = \frac{10^7 - \#\text{have no repeating digit}}{10^7}$$

$$= \frac{10^7 - 10 \cdot 9 \cdot 8 \cdot 7 \cdot \ldots \cdot 4}{10^7} = 1 - \Pr(\text{no repeated digit})$$

Other problems

Probability that a 3 character password has at least one digit?
Each character is either a digit (0-9) or a lower case letter (a-z).

overcounting

$$\frac{10\ 36\ 36 + 36\ 10\ 36 + 36\ 36\ 10}{36^3}$$

- Probability that a randomly placed pawn, bishop and knight share no row or column?

Ω = possible placement of pawn, placement of bishop, placement of knight

$|\Omega| = 64 \cdot 63 \cdot 62$

$\Pr(\omega) = \frac{1}{64 \cdot 63 \cdot 62}$

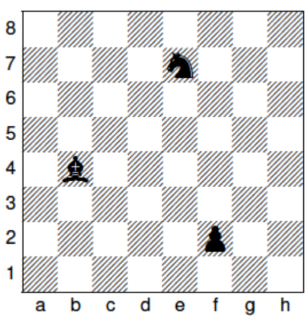
E : placements of 3 pieces s.t. none of them share a row or column

$\Pr(E) = \sum_{\omega \in E} \Pr(\omega) = \frac{|E|}{|\Omega|} = \frac{8^2 \cdot 7^2 \cdot 6^2}{64 \cdot 63 \cdot 62}$

$|E| = 64 \cdot (64 - 2 \cdot 8 + 1) \cdot (64 - 4 \cdot 8 + 4)$

$\binom{8}{3} \cdot \binom{8}{3}\, 3!\, 3!$

$8^2 \cdot 7^2 \cdot 6^2$



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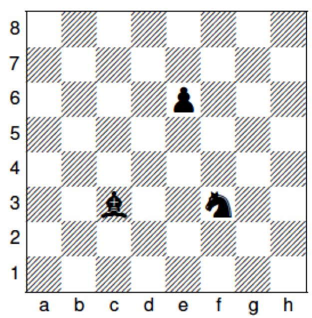


(b) invalid

Rooks on Chessboard

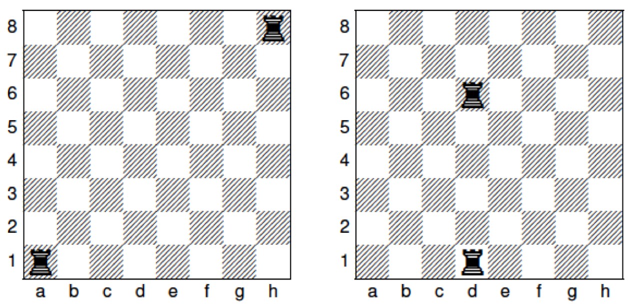
Probability that two randomly placed identical rooks don't share a row or column?

Ω: set of placements of 2 rooks $\qquad |\Omega| = \binom{64}{2}$

E: rooks are placed so that they don't share a row or column

$$|E| = \frac{8^2 \cdot 7^2}{2!}$$

$$\Pr(E) = \frac{|E|}{|\Omega|} = \frac{8^2 \cdot 7^2}{2 \cdot \binom{64}{2}}$$



(a) valid (b) invalid