Counting

First rule of counting:
Product Rule

- If S is a set of sequences of length k for which there are
  - n_1 choices for the first element of sequence
  - n_2 choices for the second element given any particular choice for first
  - n_3 choices for third given any particular choice for first and second.
  - .....
- Then $|S| = n_1 \times n_2 \times \ldots \times n_k$

52 card deck

choose ordered

seq of 5 cards.

$$52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = \frac{52!}{(52-5)!}$$

First rule of counting:
Generalized Product Rule

- If S is a set of sequences of length k for which there are
  - n_1 choices for the first element of sequence
  - n_2 choices for the second element given any particular choice for first
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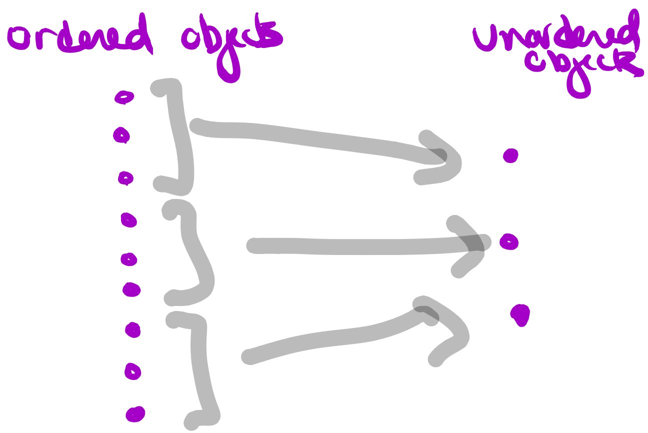
  - .....
- Then $|S| = n_1 \times n_2 \times \ldots. \times n_k$

Application: Number of ways of choosing an ordered sequence of r items out of n distinct items: $n!/(n-r)!$

$$\frac{n!}{(n-r)!}$$

Second rule of counting:

- If order doesn't matter, count *ordered objects* and then divide by the number of orderings.



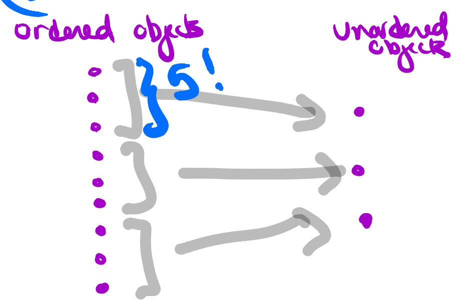
$$\#\text{unordered objects} = \frac{\#\text{ordered objects}}{3}$$

Second rule of counting:

- If order doesn't matter, count *ordered objects* and then divide by the number of orderings.

- Example: how many 5 card poker hands?

$$\frac{52!}{(52-5)!}$$



$$\underbrace{\#\text{ordered}}_{\frac{52!}{(52-5)!}} = 5!\ \#\text{unordered}$$

$$\#\text{unordered} = \frac{52!}{(52-5)!\ 5!} = \binom{52}{5}$$

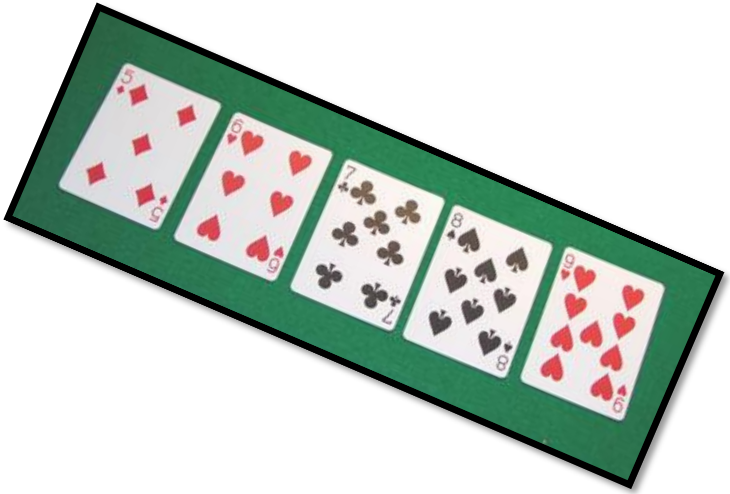
Combinations

- Number of ways to choose r unordered objects out of n distinct objects
  - A: set of ordered lists of r out of n objects
  - B: set of unordered lists of r out of n objects
  - Each ordered list maps to one unordered list.
  - Each unordered list has r! ordered lists that map to it.
  - |A| = r! |B|

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 1} = \frac{n!}{r!(n-r)!}$$

Called "n choose r"

quick review of cards



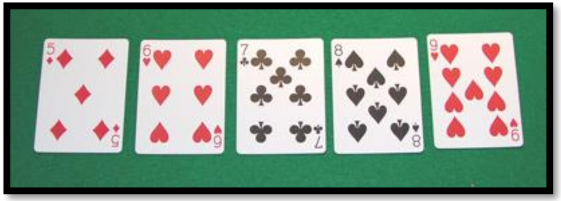
- 52 total cards
- 13 different **ranks**: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different **suits**: Hearts, Clubs, Diamonds, Spades

- How many possible 5 card hands? $\binom{52}{5}$

- A "straight" is five consecutive rank cards of any suit. How many possible straights?

$\underbrace{\text{choose rank of lowest card}}_{10}$ · $\underbrace{\text{suit for lowest} \cdots \text{suit for highest}}_{4^5}$

$10 \cdot 4^5$



- How many flushes are there?

choose suit · choose unordered set of 5 cards of that suit

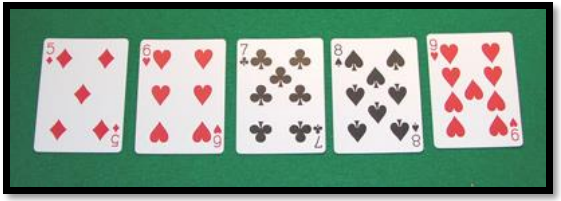
$4 \cdot \binom{13}{5}$

- How many possible 5 card hands?

$$\binom{52}{5}$$

- A "straight" is five consecutive rank cards of any suit. How many possible straights?

$$10 \cdot 4^5 = 10,240$$

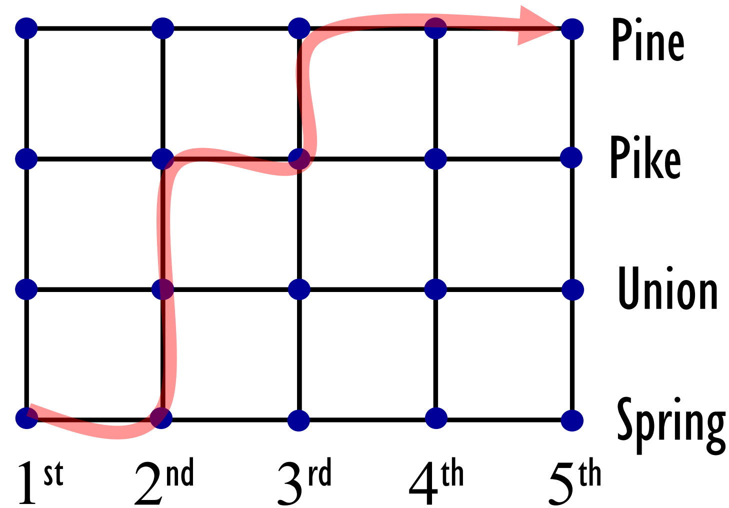


- How many flushes are there?

$$4 \cdot \binom{13}{5} = 5,148$$

How many ways to walk from 1st and Spring to 5th and Pine only going North and East?

Instead of tracing paths on the grid above, list choices. You walk 7 blocks; at each intersection choose N or E; must choose N exactly 3 times.



$$\binom{7}{3} = 35$$

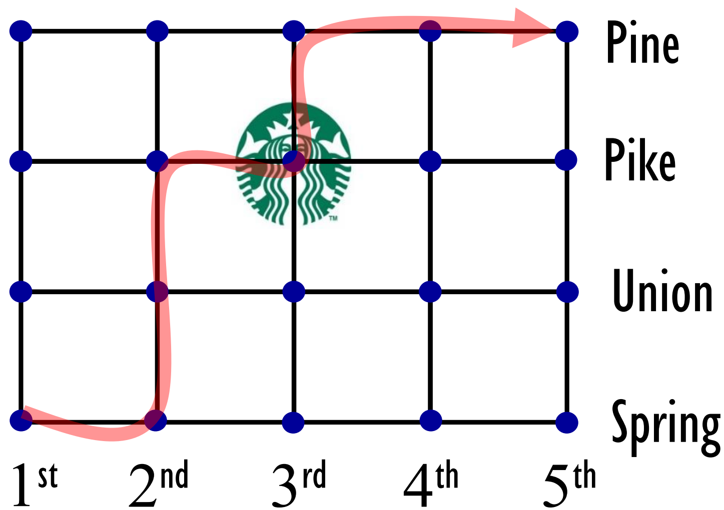
1st step

2nd

$= \binom{7}{4}$

How many ways to walk from 1^{st} and Spring to 5^{th} and Pine only going North and East, if I want to stop at Starbucks on the way?

$$\binom{4}{2} \cdot \binom{3}{1}$$



$$\binom{4}{2} \cdot \binom{3}{2}$$

$$\binom{n}{r} = \binom{n}{n-r}$$

For each object constructed it should be possible to reconstruct the unique sequence of choices that led to it!

choice 1, choice 2, ... choice k

$n_1 \cdot n_2 \cdot \cdots \cdot n_k$

For each object constructed it should be possible to reconstruct the unique sequence of choices that led to it!

Example: How many ways are there to choose a 5 card hand that contains at least 3 aces?

$$\binom{4}{3} \cdot \binom{49}{2}$$

Choose 3 aces, then choose 2 cards from remaining 49.

$$\underbrace{A\diamondsuit \quad K\heartsuit \quad A\clubsuit \quad A\spadesuit \quad A\heartsuit}_{\text{gets counted 4 times}}$$

to fix, subtract

$$3\binom{4}{4}\binom{48}{1}$$

For each object constructed it should be possible to reconstruct the unique sequence of choices that led to it!

Example: How many ways are there to choose a 5 card hand that contains at least 3 aces?

$$\cancel{\binom{4}{3} \cdot \binom{49}{2}}$$

~~Choose 3 aces, then choose 2 cards from remaining 49.~~

$$\binom{4}{3} \cdot \binom{48}{2} + \binom{4}{4} \cdot \binom{48}{1}$$

When in doubt break set up into disjoint sets you **know** how to count!

Combinations: Number of ways to choose r things from n things

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Pronounced "n choose r" aka "binomial coefficients"

E.g., $$\binom{n}{2} = \frac{n(n-1)}{2} = \Theta(n^2)$$

Many identities:

1. $$\binom{n}{r} = \binom{n}{n-r}$$
2. $$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$
3. $$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$$

of size r

② # subsets of size r that contain elt 1 + # subsets of size r that don't contain elt 1. $= \binom{n}{r}$

$\binom{n-1}{r-1}$ $\binom{n-1}{r}$

$$r\binom{n}{r} = n\binom{n-1}{r-1}$$

choose a team of r players out of n & captain for the team.

Combinatorial proof

- Let S be a set of objects.
- Show how to count the set one way => N
- Show how to count the set another way => M

- Therefore N=M

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Many identities:

$$\binom{n}{r} = \binom{n}{n-r}$$ ← by symmetry of definition

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$ ← 1st object either in or out

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$$ ← team + captain

$\forall x, y$

n

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \underline{x^k y^{n-k}}}$$

Proof 1: Induction …

Proof 2: Counting

$x^2 y^{n-2}$

(x+y) • (x+y) • (x+y) • … • (x+y)

Pick either x or y from first factor

Pick either x or y from second factor

…

Pick either x or y from nth factor

$(x+y)(x+y)(x+y)$

$x^3 + 3x^2y + 3xy^2 + y^3$

$\binom{3}{0} y^3 + \binom{3}{1} x^1 y^2 + \binom{3}{2} x^2 y + \binom{3}{3} x^3$

How many ways to get exactly k x's?

an identity with binomial coefficients

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

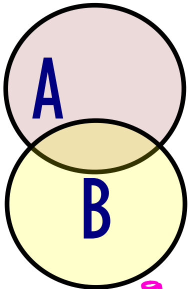
Proof:

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

$x=1, y=1$

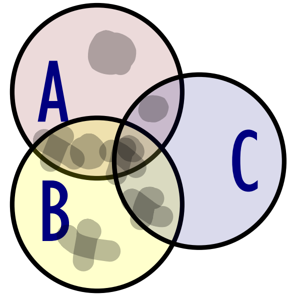
$$\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n$$

inclusion/exclusion principle



7 in first pos

7 in 2nd pos



$10^9 \qquad 10^9 \qquad 10^8$

$$|A\cup B| = |A| + |B| - |A\cap B|$$

10 digit #'s that have 7 in 1st position or in 2nd position

7 7 _ _ _ _ _ _ _ _

$$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|$$

General: + singles - pairs + triples - quads + ...

pigeonhole principle

If there are **n** pigeons in **k** holes and **n > k**, then **some hole contains more than one pigeon.**

If there are **n** pigeons in **k** holes and **n > k**, then **some hole contains more than one pigeon.**

More precisely, some hole contains at least $\lceil n/k \rceil$ pigeons.

To solve a PHP problem:

1. Define the pigeons
2. Define the pigeonholes
3. Define the mapping of pigeons to pigeonholes

If there are **n** pigeons in **k** holes and **n > k**, then **some hole contains more than one pigeon.**

More precisely, some hole contains at least $\lceil n/k \rceil$ pigeons.

Use the PHP to prove that n a room of 500 people, there are two people who share a birthday.

Pigeons:

Pigeonholes:

Rule for assigning pigeon to pigeonhole:

Use Pigeonhole Principle to show that…

- In every set of 100 numbers, there are two whose difference is a multiple of 37.

Pigeons:

Pigeonholes:

Rule for assigning pigeon to pigeonhole:

So far

- Product Rule
- Sum Rule
- Inclusion-exclusion
- Permutations/combinations
- Binomial Theorem
- Combinatorial proofs
- Pigeonhole principle

Doughnuts

- You go to Top Pot to buy a dozen doughnuts. Your choices today are
  - Chocolate
  - Lemon-filled
  - Sugar
  - Glazed
  - Plain
- How many ways to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

Bijection Rule

- Count one set by counting another.
- Example:
  - A: all ways to select a dozen doughnuts when five varieties are available.
  - B: all 16 bit sequences with exactly 4 ones

$$\underbrace{0\ 0}_{\text{chocolate}} \quad \underbrace{\qquad}_{\text{lemon-filled}} \quad \underbrace{0\ 0\ 0\ 0\ 0\ 0}_{\text{sugar}} \quad \underbrace{0\ 0}_{\text{glazed}} \quad \underbrace{0\ 0}_{\text{plain}}$$

Bijection between A and B

– A: all ways to select a dozen doughuts when five varieties are available.

– B: all 16 bit sequences with exactly 4 ones

$$\underbrace{0\;0}_{\text{chocolate}} \qquad \underbrace{\qquad}_{\text{lemon-filled}} \qquad \underbrace{0\;0\;0\;0\;0\;0}_{\text{sugar}} \qquad \underbrace{0\;0}_{\text{glazed}} \qquad \underbrace{0\;0}_{\text{plain}}$$

$$\underbrace{0\;0}_{\text{chocolate}} \quad 1 \quad \underbrace{\qquad}_{\text{lemon-filled}} \quad 1 \quad \underbrace{0\;0\;0\;0\;0\;0}_{\text{sugar}} \quad 1 \quad \underbrace{0\;0}_{\text{glazed}} \quad 1 \quad \underbrace{0\;0}_{\text{plain}}$$

$$0011000000100100\,.$$

Bijection between A and B

- A: all ways to select a dozen doughnuts when five varieties are available.
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$$\underbrace{0\,0}_{\text{chocolate}}\quad\underbrace{}_{\text{lemon-filled}}\quad\underbrace{0\,0\,0\,0\,0\,0}_{\text{sugar}}\quad\underbrace{0\,0}_{\text{glazed}}\quad\underbrace{0\,0}_{\text{plain}}$$

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$$0011000000100100\,.$$

$$\binom{16}{4}$$

Mapping from doughnuts to bit strings

c chocolate, l lemon-filled, s sugar, g glazed, and p plain

to the sequence:

$$\underbrace{0\ldots0}_{c} \quad 1 \quad \underbrace{0\ldots0}_{l} \quad 1 \quad \underbrace{0\ldots0}_{s} \quad 1 \quad \underbrace{0\ldots0}_{g} \quad 1 \quad \underbrace{0\ldots0}_{p}$$

Other problems

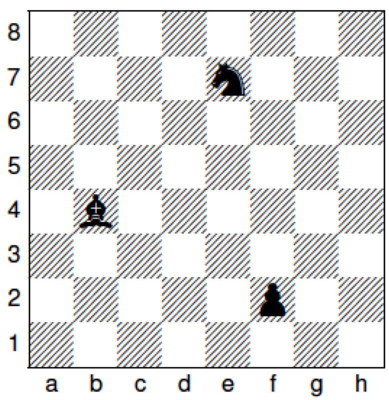
of 7 digit numbers (decimal) with at least one repeating digit? (allowed to have leading zeros).

# of 3 character password with at least one digit
each character either digit 0-9 or letter a-z.

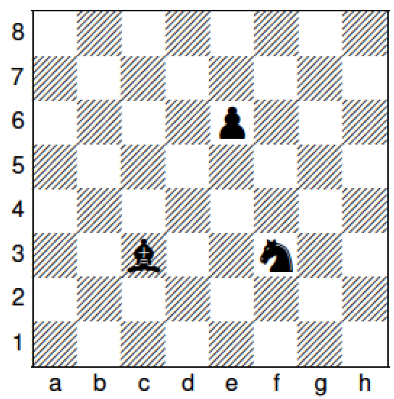
10 36 36 + 36 10 36 + 36 36 10

8 by 8 chessboard

- How many ways to place a pawn, bishop and knight so that none are in same row or column?



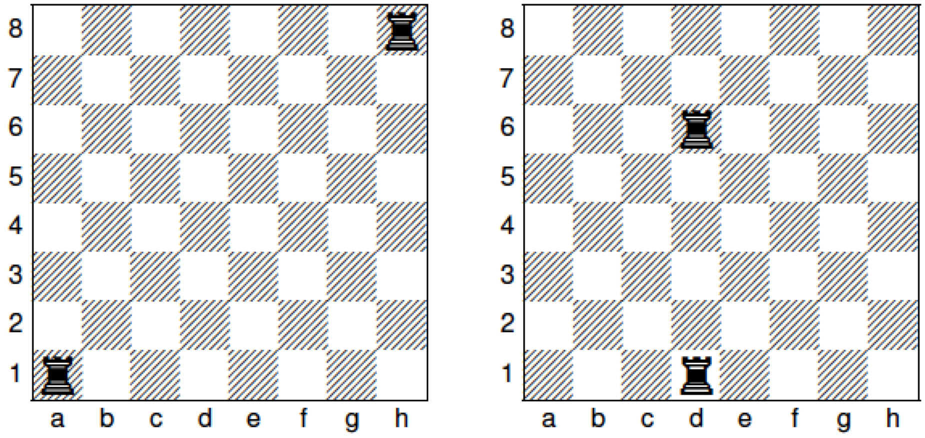
(a) valid



(b) invalid

Rooks on Chessboard

- Number of ways to place 2 identical rooks on a chessboard so that they don't share a row or column.



(a) valid (b) invalid

Buying 2 dozen bagels

- Choosing from 3 varieties:
  - Plain
  - Garlic
  - Pumpernickel
- How many ways to grab 2 dozen if you want at least 3 of each type and bagels of the same type are indistinguishable.

- Must get 3 of each type, so have 15 left over to choose.
- Bijection with bit strings of length 17 with 2 1s.

Lessons

- Solve the same problem in different ways!
- If needed, break sets up into disjoint subsets that you know for sure how to count.
- Have in mind a sequence of choices that produces the objects you are trying to count. (Usually there are many possibilities.)
- Once you specify the sequence of choices you are making to construct the objects, make sure that given the result, you can tell exactly what choice was made at each step!