Note: Linearity is special!

It is *not* true in general that

$$E[X \cdot Y] = E[X] \cdot E[Y]$$

$$E[X^2] = E[X]^2$$

$$E[X/Y] = E[X] / E[Y]$$

$$E[\operatorname{asinh}(X)] = \operatorname{asinh}(E[X])$$

$$\vdots$$

# variance

$$Var(X) = E\left[(X - E(X))^2\right] = E[X^2] - (E(X))^2$$

$\sigma(X)$ - standard deviation $= \sqrt{Var(X)}$

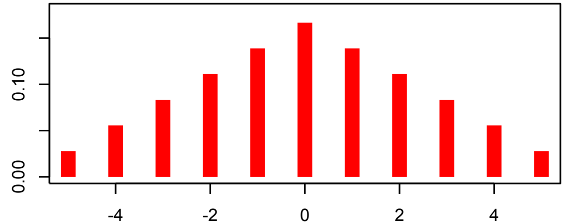
$$Var(aX + b) = a^2 Var(X)$$

$a, b$ are constants

# more variance examples

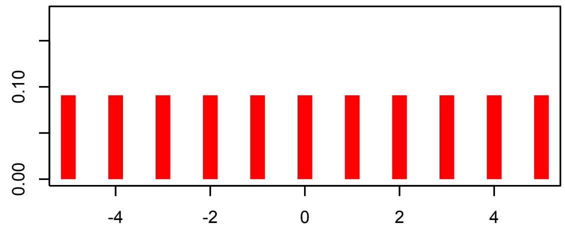
$p_{X_1}(x) = \Pr(X_1 = x)$

$X_1$



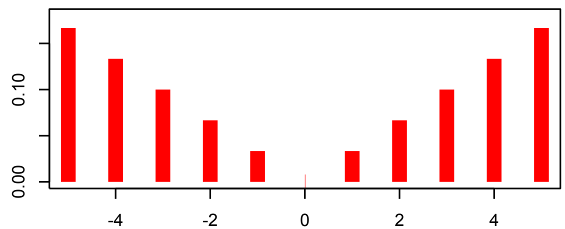

$\sigma^2 = 5.83$

$X_2$



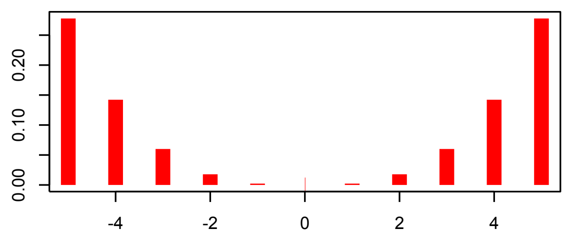

$\sigma^2 = 10$

$X_3$




$\sigma^2 = 15$

$X_4$




$\sigma^2 = 19.7$

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2$$

---

$$\mathrm{Var}(X+Y) \neq \mathrm{Var}(X) + \mathrm{Var}(Y) \quad \text{in general}$$

Example 1

$$X = \begin{cases} 1 & \frac{1}{2} \\ -1 & \frac{1}{2} \end{cases} \qquad E(X) = 0$$

$$\mathrm{Var}(X) = E(X^2) = 1$$

$$Y = -X \qquad E(Y) = 0$$

$$\mathrm{Var}(Y) = 1$$

$$\mathrm{Var}(X) + \mathrm{Var}(Y) = 2 \neq \mathrm{Var}(X+Y) = 0$$

~~Example 2~~

$$\mathrm{Var}(X+X) = \mathrm{Var}(2X) = 2^2\,\mathrm{Var}(X) = 4\,\mathrm{Var}(X)$$

$$\neq \mathrm{Var}(X) + \mathrm{Var}(X)$$

# Random variables & independence.

A r.v. $X$ and an event $E$ are independent if

$$\forall x \quad \Pr(X=x \cap E) = \Pr(X=x)\Pr(E)$$

2 r.v.'s $X$ & $Y$ are independent if

$$\forall x \,\forall y \quad \Pr(X=x \cap Y=y) = \Pr(X=x)\Pr(Y=y)$$

Flip a fair coin independently $2n$ times

$Z$: # heads in $2n$ tosses,

$X$: # heads in first $n$ tosses

$Y$: # heads in 2nd $n$ tosses

**X & Y indep**

$$\Pr(X=i) = \frac{\binom{n}{i}}{2^n} = \binom{n}{i}\left(\tfrac{1}{2}\right)^i\left(\tfrac{1}{2}\right)^{n-i}$$

$$\Pr(Y=j) = \frac{\binom{n}{j}}{2^n}$$

$$\Pr(X=i \cap Y=j) = \binom{n}{i}\binom{n}{j}\frac{1}{2^{2n}}$$

**X & Z not indep**

$$\Pr(X=x \cap Z=z) \neq \Pr(X=x)\Pr(Z=z)$$

$x=0 \implies Z \le n$

$z=2n$

**Thm:** If $X$ & $Y$ are independent, then $E(X \cdot Y) = E(X)E(Y)$

$$E(X \cdot Y) = \sum_{a \in \mathrm{Range}(X)} \sum_{b \in \mathrm{Range}(Y)} a \cdot b \Pr(X=a \cap Y=b)$$

$$= \sum_a \sum_b a \cdot b \Pr(X=a)\Pr(Y=b)$$

by indep of $X$ & $Y$

$$= \sum_a a \Pr(X=a) \cancel{\sum_b} \cancel{b} \Pr(Y=b)$$

$$= E(X) \cdot E(Y)$$

| | $X$ | | $Y$ | |
|---|---|---|---|---|
| $\frac{3}{4}$ | 1 | | 2 | $\frac{1}{3}$ |
| $\frac{1}{4}$ | 2 | | 7 | $\frac{1}{3}$ |
| | | | 9 | $\frac{1}{3}$ |

Theorem: If X & Y are *independent*, then

Var[X+Y] = Var[X]+Var[Y]

Proof:

$$Var[X+Y]$$

$$= E[(X+Y)^2] - (E[X+Y])^2$$

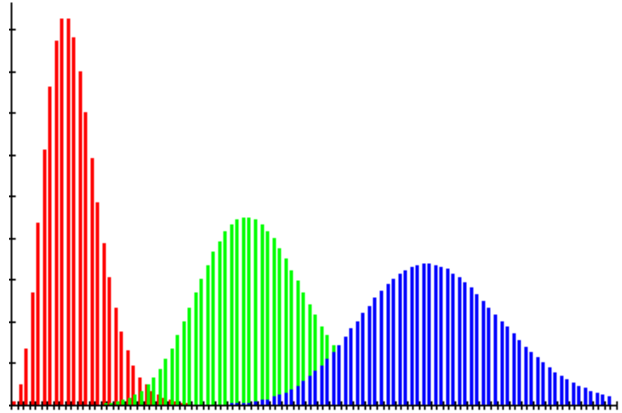
$$= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2$$

$$= E[X^2] + 2E[XY] + E[Y^2] - ((E[X])^2 + 2E[X]E[Y] + (E[Y])^2)$$

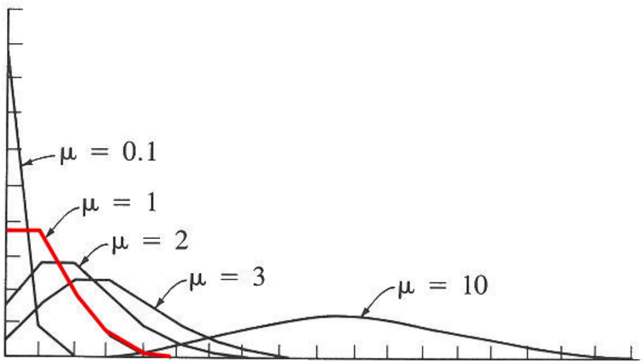
$$= E[X^2] - (E[X])^2 + E[Y^2] - (E[Y])^2 + 2(E[XY] - E[X]E[Y])$$

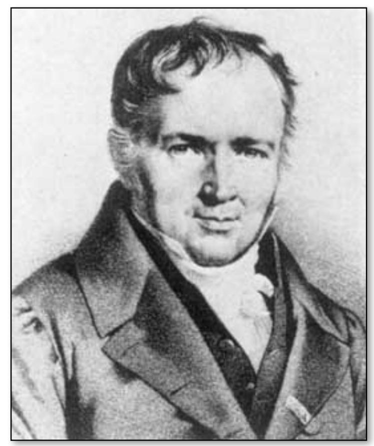
$$= Var[X] + Var[Y] + 2(E[X]E[Y] - E[X]E[Y])$$

$$= Var[X] + Var[Y]$$



# a zoo of (discrete) random variables



A discrete random variable $X$ equally likely to take any (integer) value between integers $a$ and $b$, inclusive, is *uniform*.

Toss die $1, 2, \ldots, 6$

$a=1, b=6$

$a, a+1, a+2, \ldots, b$

*Notation:* $\mathrm{Unif}(a,b)$

*Probability mass function:* $\Pr(X=i) = \begin{cases} \frac{1}{b-a+1} & i \in \{a, \ldots, b\} \\ 0 & i \neq \{a, \ldots, b\} \end{cases}$

*Mean:* $\frac{a+b}{2}$

*Variance:* $\frac{(b-a)(b-a+2)}{12}$

A discrete random variable $X$ equally likely to take any (integer) value between integers $a$ and $b$, inclusive, is *uniform*.

*Notation:* $X \sim \text{Unif}(a,b)$

*Probability:* $$P(X = i) = \frac{1}{b - a + 1}$$

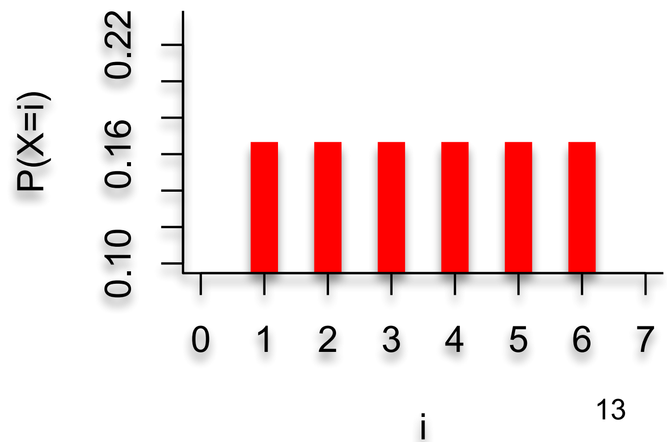
*Mean, Variance:* $$E[X] = \frac{a + b}{2}, \text{Var}[X] = \frac{(b - a)(b - a + 2)}{12}$$

*Example:* value shown on one roll of a fair die is Unif(1,6):

$P(X=i) = 1/6$

$E[X] = 7/2$

$\text{Var}[X] = 35/12$



coin tosses with p of coming up H

In a series $X_1, X_2, \ldots$ of ~~Bernoulli~~ trials with success probability p, let Y be the index of the first success, i.e.,

$$X_1 = X_2 = \ldots = X_{Y-1} = 0 \ \&\ X_Y = 1$$

Then Y is a *geometric* random variable with parameter p.

$X \sim \text{Geo}(p)$

Examples:

- Number of coin flips until first head
- Number of blind guesses on SAT until I get one right
- Number of darts thrown until you hit a bullseye
- Number of random probes into hash table until empty slot
- Number of wild guesses at a password until you hit it

TTTTH

$(1-p)^4 p$

*Probability mass function:*

$$\Pr(X=i) = \begin{cases} (1-p)^{i-1}p & i=1,2,\ldots \\ 0 & \text{o.w.} \end{cases}$$

*Mean:* $\frac{1}{p}$

*Variance:* $\frac{(1-p)}{p^2}$

In a series $X_1, X_2, \ldots$ of Bernoulli trials with success probability p, let Y be the index of the first success, i.e.,

$$X_1 = X_2 = \ldots = X_{Y-1} = 0 \ \& \ X_Y = 1$$

Then Y is a *geometric* random variable with parameter p.

Examples:

Number of coin flips until first head

Number of blind guesses on SAT until I get one right

Number of darts thrown until you hit a bullseye

Number of random probes into hash table until empty slot

Number of wild guesses at a password until you hit it

$P(Y=k) = (1-p)^{k-1}p;$

Mean $1/p$; Variance $(1-p)/p^2$

indicator r.v.

An experiment results in "Success" or "Failure"

X is an *indicator random variable* (1 = success, 0 = failure)

P(X=1) = p and P(X=0) = 1-p

X is called a *Bernoulli* random variable: X ~ Ber(p)

| $X^2$ | $X$ | Pr |
|---|---|---|
| 1 | 1 | p |
| 0 | 0 | 1-p |

expectation

*Mean:* $p$

*Variance:* $p - p^2$

$= p(1-p)$

$$E(X) = \sum_{x \in Range(X)} x \Pr(X=x)$$

$$Var(X) = E(X^2) - E(X)^2$$

$= p$

An experiment results in "Success" or "Failure"

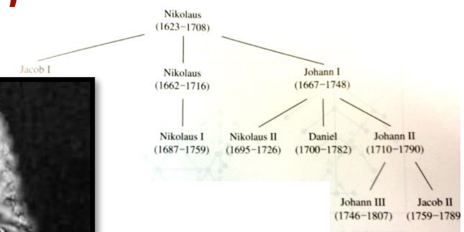
X is an *indicator random variable* (1 = success, 0 = failure)

$P(X=1) = p$ and $P(X=0) = 1-p$

X is called a *Bernoulli* random variable: $X \sim \mathrm{Ber}(p)$

$E[X] = E[X^2] = p$

$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$




Examples:

- coin flip
- random binary digit
- whether a disk drive crashed

Jacob (aka James, Jacques) Bernoulli, 1654 – 1705