Random variables

A random variable (r.v.) X on a prob space (Ω, P) is a function $X : \Omega \to \mathbb{R}$

$\{X=a\}$ is an event $= \{\omega \mid X(\omega)=a\}$

The expectation or expected value or mean of a r.v. is defined as

$$\boxed{E(X) = \sum_{k \in Range(X)} k \, Pr(X=k) = \sum_{\omega \in \Omega} X(\omega) \, Pr(\omega)}$$

Expectation of a function of a r.v.

$Y = f(X)$ $\quad f : \mathbb{R} \to \mathbb{R}$

$$E(Y) = \sum_{k \in Range(Y)} k \, Pr(Y=k) = \boxed{\sum_{j \in Range(X)} f(j) \, Pr(X=j)} = \sum_{\omega \in \Omega} f(X(\omega)) \, Pr(\omega)$$

Linearity of expectation

Thm: $\forall$ 2 random vars X & Y defined on same prob space

$$E(X+Y) = E(X) + E(Y)$$

Corollary (by induction)

If $X = X_1 + X_2 + \cdots + X_n$

then

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

$$E(cX) = cE(X)$$

Variance of a r.v.

measure of deviation of r.v. from its expected value.

$$Var(X) = E\left((X - E(X))^2\right)$$

standard deviation of r.v. $\sigma(X) \triangleq \sqrt{Var(X)}$

Game 1: toss a fair coin
if comes up H's you pay me \$1
o.w. I pay you a \$1

Game 2: toss a fair coin
H's $\Rightarrow$ you pay me \$1000
T's $\Rightarrow$ I pay you \$1000

X: my winnings in this game.

		X	$(X-E(X))^2 = X^2$
$\frac{1}{2}$	H	1	1
$\frac{1}{2}$	T	-1	1

$E(X) = 0$

$Var(X) = 1$

X	X^2
1000	1,000,000
-1000	1,000,000

$E(X) = 0$

$Var(X) = 1,000,000$

3 homeworks returned to n students; shuffled randomly

X: # students that receive their own homework
takes values $\{0,1,3\}$

$Pr(\omega)$	ω	$X(\omega)$	X^2	$(X-1)^2$
$1/6$	1 2 3	3	9	4
$1/6$	1 3 2	1	1	0
$1/6$	2 1 3	1	1	0
$1/6$	3 2 1	1	0	0
$1/6$	3 1 2	0	0	1
$1/6$	2 3 1	0	0	1

$Var(X)$
$= 4 \cdot Pr(X=3)$
$+ 0 \cdot Pr(X=1)$
$+ 1 \cdot Pr(X=0)$
$= \frac{4}{6} + \frac{1}{3} = 1$

$$E(X) = 1$$

X, $\mu = E(X)$

Claim: $Var(X) = E\left((X-\mu)^2\right) = E(X^2) - \mu^2$

$E(c) = c$

Pf: $E\left((X-\mu)^2\right) = E\left[X^2 - 2\mu X + \mu^2\right]$

lin of exp $= E(X^2) - E(2\mu X) + \mu^2$

$= E(X^2) - 2\mu E(X) + \mu^2$

$= E(X^2) - 2\mu^2 + \mu^2$

$= E(X^2) - \mu^2$

		X	X^2	X_1	X_2
$\frac{1}{4}$	HH	2	4	1	1
	HT	1	1	1	0
⋮	TH	1	1	0	1
$\frac{1}{4}$	TT	0	0	0	0

n indep coin tosses, each with prob p of coming up H's.

X: # of heads.

Binomial r.v. : defined n, p

$$X = X_1 + X_2 + \dots + X_n$$

$$X_i = \begin{cases} 1 & i^{th} \text{ coin toss was H's} \\ 0 & \text{o.w.} \end{cases}$$

indicator r.v.

Bernoulli r.v. param p.

$= Pr(X_i = 1)$

$E(X_i) = Pr(X_i = 1)$
$= p$

$$E(X) = np$$

$$Var(X) = E(X^2) - (np)^2 \quad \text{by claim.}$$

X_i	X_i^2
1	1
1	1
1	1
0	0
0	0
0	0

$$E(X^2) = E\Big(\overbrace{(X_1 + X_2 + \dots + X_n)}^{X}\overbrace{(X_1 + X_2 + \dots + X_n)}^{X}\Big)$$

$$= E\Big[\sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n} \sum_{j \neq i} X_i X_j\Big]$$

$$= \sum_{i=1}^{n} \underbrace{E(X_i^2)}_{p} + \sum_{i=1}^{n} \sum_{j \neq i} \underbrace{E(X_i X_j)}_{p^2}$$

linearity of expectation

$E(X_i X_j)$
$= 1 \cdot \boxed{Pr(X_i X_j = 1)}$
$+ 0 \cdot Pr(X_i X_j = 0)$

$Pr(X_i = 1 \text{ and } X_j = 1)$
$\underset{\text{independence of coin tosses}}{=} Pr(X_i = 1)\, Pr(X_j = 1)$
$= p^2$

$$= np + n(n-1)p^2$$

$E(X_i^2) = 1 \cdot \underbrace{Pr(X_i^2 = 1)}_{Pr(X_i = 1) = p} + 0 \cdot Pr(X_i^2 = 0)$

$$Var(X) = E(X^2) - \mu^2$$

$$= np + n(n-1)p^2 - n^2p^2$$

$$= np - np^2 = np(1-p)$$

Because $\quad Var(X) = E\left((X-\mu)^2\right)$

$Var(X)$ always ≥ 0

$$\sigma(X) = \sqrt{Var(X)}$$

$$\cancel{E\left((X-\mu)\right) = 0}$$

Claim: $\quad Var(\overbrace{aX+b}^{Z}) = a^2 Var(X)$ $\quad$ a & b constants.

Pf

$$Var(aX+b) = E\Big[\big(\underbrace{aX+b}_{Z} \;\; \underbrace{-a\mu-b}_{E(Z)}\big)^2\Big]$$

$E(aY+b) = a\mu+b$

$Var(Y) = E\left((Y-\mu)^2\right)$

$$= E\Big[\big[a(X-\mu)\big]^2\Big] = E\left[a^2 (X-\mu)^2\right]$$

$$= a^2 \underbrace{E\left((X-\mu)^2\right)}_{Var(X)}$$

$$= a^2 Var(X)$$

flip 100 fair coins

$$Z = \underbrace{\#\text{heads}}_{X} - \underbrace{\#\text{tails}}_{Y} = X - (100 - X)$$

$$= 2X - 100$$

$$E(Z) = E(X) - E(Y)$$

$$= 50 - 50 = 0$$

$$Var(Z) = Var(2X - 100) = 2^2 \underbrace{Var(X)}_{\frac{100}{4}}$$

apply claim

$a = 2$

$b = -100$

$np(1-p)$

$$= 100$$