Random variables

A random variable (r.v.) $X$ on a prob space $(\Omega, \mathcal{F}, P)$ is a function $X: \Omega \rightarrow \mathbb{R}$

$$\{X = a\} \text{ is an event} = \{w \in \Omega \mid X(w) = a\}$$

The expectation or expected value or mean of a r.v. is defined as

$$E(X) = \sum_{k \in \text{range}(X)} k \cdot \Pr(X = k) = \sum_{w \in \Omega} X(w) \cdot \Pr(w)$$

Expectation of a function of a r.v.

$Y = f(X)$, $f: \mathbb{R} \rightarrow \mathbb{R}$

$$E(Y) = \sum_{k \in \text{range}(Y)} k \cdot \Pr(Y = k) = \sum_{j \in \text{range}(X)} f(j) \cdot \Pr(X = j) = \sum_{w \in \Omega} f(X(w)) \cdot \Pr(w)$$
**Linearity of expectation**

**Theorem:** For any two random variables $X$ and $Y$ defined on same prob space, $E(X+Y) = E(X) + E(Y)$.

**Corollary (by induction):** If $X=X_1+X_2+\ldots+X_n$, then $E(X) = E(X_1) + E(X_2) + \ldots + E(X_n)$. 

$E(cX) = cE(X)$
Variance of a r.v.

measures deviation of r.v. from its expected value.

\[ \text{Var}(X) = E \left( (X - E(X))^2 \right) \]

standard deviation of r.v. \( \sigma(X) = \sqrt{\text{Var}(X)} \)

Game 1: toss a coin
- if comes up H's you pay me $1
- o.w. I pay you a $1

\( X \): my winnings in this game.

<table>
<thead>
<tr>
<th>( X )</th>
<th>( (X - E(X))^2 = X^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1</td>
</tr>
<tr>
<td>T</td>
<td>-1</td>
</tr>
</tbody>
</table>

\[ E(X) = 0 \]
\[ \text{Var}(X) = 1 \]

Game 2: toss a fair coin
- H's \( \Rightarrow \) you pay me $1000
- T's \( \Rightarrow \) I pay you $1000

\[ X \quad X^2 \]
\[ 1000 \quad 1,000,000 \]
\[ -1000 \quad 1,000,000 \]

\[ E(X) = 0 \]
\[ \text{Var}(X) = 1,000,000 \]
3 homeworks returned to n students; shuffled randomly

\( X: \) # students that receive their own homework

<table>
<thead>
<tr>
<th>( \text{Pr}(w) )</th>
<th>( x(w) )</th>
<th>( x^2 )</th>
<th>( (x-1)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{6} )</td>
<td>123</td>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>( \frac{1}{6} )</td>
<td>132</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{1}{6} )</td>
<td>213</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \frac{1}{6} )</td>
<td>321</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{1}{6} )</td>
<td>312</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \frac{1}{6} )</td>
<td>231</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\[ \text{E}(X) = 1 \]

Claim: \( \text{Var}(X) = \text{E}((X-\mu)^2) = \text{E}(x^2) - \mu^2 \)

\[ \begin{align*}
\text{Var}(X) &= \text{E}((X-\mu)^2) \\
&= \text{E}(x^2) - \mu^2 \\
&= \text{E}(x^2) - 2\mu \text{E}(x) + \mu^2 \\
&= \text{E}(x^2) - 2\mu^2 + \mu^2 \\
&= \text{E}(x^2) - \mu^2
\end{align*} \]

\[ X, \mu = \text{E}(x) \]

\[ \text{Var}(X) = \text{E}(x^2) - \mu^2 \]

\[ \text{E}(x^2) = \mu^2 \]

\[ \text{Var}(X) = \text{E}(x^2) - \mu^2 \]

\[ \text{E}(x^2) = \mu^2 \]

\[ \text{Var}(X) = \text{E}(x^2) - \mu^2 \]

\[ \text{E}(x^2) = \mu^2 \]

\[ \text{Var}(X) = \text{E}(x^2) - \mu^2 \]

\[ \text{E}(x^2) = \mu^2 \]

\[ \text{Var}(X) = \text{E}(x^2) - \mu^2 \]

\[ \text{E}(x^2) = \mu^2 \]

\[ \text{Var}(X) = \text{E}(x^2) - \mu^2 \]

\[ \text{E}(x^2) = \mu^2 \]

\[ \text{Var}(X) = \text{E}(x^2) - \mu^2 \]

\[ \text{E}(x^2) = \mu^2 \]

\[ \text{Var}(X) = \text{E}(x^2) - \mu^2 \]

\[ \text{E}(x^2) = \mu^2 \]
n independent coin tosses, each with prob \( p \) of coming up H's.

\( X \): \# of heads.

**Binomial r.v.** defined \( n, p \)

\[
X = X_1 + X_2 + \ldots + X_n
\]

\[
X_i = \begin{cases} 
1 & \text{if coin toss was H's} \\
0 & \text{o.w.}
\end{cases}
\]

**Indicator r.v.**

Borellii r.v., param \( p \) = \( \Pr(X_i = 1) \)

\[E(X_i) = \Pr(X_i = 1) = p\]

**Van(X)** = \( E(X^2) - (np)^2 \) by claim.

\[
E(X^2) = E\left( \sum_{i=1}^{n} X_i^2 \right)
\]

\[
= E\left[ \sum_{i=1}^{n} X_i^2 \right] + \sum_{i=1}^{n} \sum_{j \neq i} E(X_i X_j)
\]

\[
= \sum_{i=1}^{n} E(X_i^2) + \sum_{i=1}^{n} \sum_{j \neq i} E(X_i X_j)
\]

\[
= np + n(n-1)p^2
\]

**Van(X)** = \( E(X^2) - \mu^2 \)

\[
= np + n(n-1)p^2 - np^2
\]

\[
= np - np^2 = np(1-p)
\]
Because \[ \text{Var}(X) = \mathbb{E}\left((X-\mu)^2\right) \]
\[ \text{Var}(X) \text{ always} \geq 0 \]

\[ \sigma(X) = \sqrt{\text{Var}(X)} \]

\[ \mathbb{E}\left((X-\mu)^2\right) = 0 \]

Claim: \[ \text{Var}(aX+b) = a^2 \text{Var}(X) \]

\[ a \text{ and } b \text{ constants.} \]

Proof:
\[ \text{Var}(aX+b) = \mathbb{E}\left[\left(\frac{aX+b-a\mu-b}{E(Z)}\right)^2\right] \]
\[ = E\left[\frac{a(X-M)^2}{E(Z)}\right] = E\left[\frac{a^2}{E(Z)}(X-\mu)^2\right] \]
\[ = a^2 \cdot E\left(\frac{(X-\mu)^2}{\text{Var}(X)}\right) \]
\[ = a^2 \text{Var}(X) \]
A fair coin is flipped 100 times.

Let $Z = \frac{\# \text{ heads}}{X} - \frac{\# \text{ tails}}{Y} = X - (100 - X) = 2X - 100$

$$E(Z) = E(X) - E(Y) = 50 - 50 = 0$$

$$\text{Var}(Z) = \text{Var}(2X - 100) = \frac{\text{Var}(X)}{4}$$

Applying the claim:

$$a = 2$$
$$b = -100$$

$$= 150$$