$\boxed{\text{Zoo of r.v.'s}}$

Bernoulli (p) $\quad Z \sim Ber(p)$ $\quad Z = \begin{cases} 1 & \text{w. prob } p \\ 0 & 1-p \end{cases}$

has dist'n of

$E(Z) = p \qquad Var(Z) = p(1-p)$

$\boxed{\text{Binomial } (n, p)}$ $\quad X \sim Bin(n,p)$

[# of H's in n indep coin tosses each w/ prob p of coming up H's]

$X = X_1 + X_2 + \dots + X_n$

$X_i \sim Ber(p)$

X_i's are mutually indep.

X takes values $0, 1, 2, \dots, n$

$$Pr(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$$

$$\sum_{i=0}^{n} Pr(X=i) = 1$$

$$E(X) = np$$

$$Var(X) = np(1-p)$$

$$Var(X) \underset{\uparrow}{=} Var(X_1) + Var(X_2) + \dots + Var(X_n)$$

because X_i's are mutually indep.

Aside

Let Z is a r.v.

$$\sum_{a \in Range(Z)} Pr(Z=a) = 1$$

C.D.F. $\quad F_X(a) = Pr(X \leq a)$

monotone fn from 0 to 1

$Var(X^2 + Y^2) = Var(X^2) + Var(Y^2)$

X & Y are indep. $\Rightarrow$ X^2 and Y^2 are indep.

Ex: sending n bit string over a network

each bit corrupted independently with prob p.

\# corrupted bits $\sim \text{Bin}(n,p)$

$n = 10^6$ $\quad p = 10^{-5}$ $\quad$ exp # of corrupted bits $= 10$

n very large $\to \infty$ $\quad np$ const.

Prob event in any given tiny interval p. $\quad$ # intervals is n

events indep $\quad \boxed{np = \lambda}$

$$\Pr(\underbrace{X}_{\#\text{ events}} = 0) = (1-p)^n \qquad e^{-x} = 1 - x + \frac{x^2}{2} \ldots$$

$$= (e^{-p})^n = e^{-pn} = e^{-\lambda}$$

if x very tiny $\quad \boxed{e^{-x} \approx 1 - x}$

$$\Pr(X=1) = \binom{n}{1} p^1 (1-p)^{n-1} = np(1-p)^{n-1} \approx npe^{-p(n-1)}$$

$$\approx npe^{-pn}$$

$$= \lambda e^{-\lambda}$$

$$\Pr(X=2) = \binom{n}{2} p^2 (1-p)^{n-2} = \frac{\lambda^2}{2} e^{-\lambda}$$

$$\boxed{X \sim \text{Bin}(n,p): \quad \lim_{\substack{n\to\infty \\ np \to \lambda}} \Pr(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}}$$

$$\boxed{Z \sim \text{Poisson}(\lambda) \qquad Z \text{ takes } 0,1,2,\ldots \qquad \Pr(Z=k) = e^{-\lambda}\frac{\lambda^k}{k!}}$$

$$1 = \sum_{k=0}^{\infty} \Pr(Z=k) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{\text{T.S of } e^{\lambda}}$$

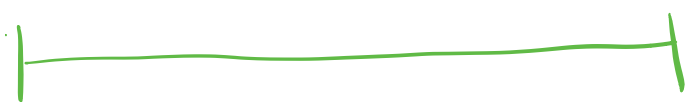
$$= 1$$

$$\boxed{E(Z) = \lambda}$$

$$= \sum_{k=0}^{\infty} k \Pr(Z=k)$$

$$= \sum_{k=0}^{\infty} k\, e^{-\lambda}\frac{\lambda^k}{k!}$$

$$\boxed{\mathrm{Var}(Z) = \lambda}$$



events happen at rate of λ/unit time

1 time unit

Conditional expectation of X given event A

$$E(X|A) = \sum_{a \in Range(X)} a \boxed{Pr(X=a|A)}$$

Law of Total Expectation.

$$E(X) = E(X|A)Pr(A) + E(X|\bar{A})Pr(\bar{A})$$

Pf

$$E(X) = \sum_{a \in Range(X)} a \underbrace{Pr(X=x)}_{\text{Law of total prob}}$$

$E(X|A) \overset{?}{=}$

$$= \sum_{a \in Range(X)} a \left[Pr(X=x|A)Pr(A) + Pr(X=x|\bar{A})Pr(\bar{A}) \right]$$