$X \sim \text{Bin}(n,p)$

$$\boxed{\lim_{\substack{n\to\infty \\ np\to\lambda}} \Pr(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}}$$

$$1 = \sum_{k=0}^{\infty} \Pr(Z=k) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{\text{T.S. of } e^{\lambda}}$$

$$= 1$$

$Z \sim$ **Poisson** (λ) $\quad$ Z takes $0, 1, 2, \ldots$

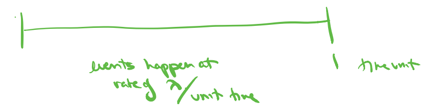
$$\Pr(Z=k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

$\boxed{E(Z) = \lambda}$

$$= \sum_{k=0}^{\infty} k \Pr(Z=k)$$

$$= \sum_{k=0}^{\infty} k\, e^{-\lambda}\frac{\lambda^k}{k!}$$

$\boxed{\text{Var}(Z) = \lambda}$



$X \sim \text{Poi}(\lambda_1)$ $\quad$ $Y \sim \text{Poi}(\lambda_2)$ $\quad$ Show $X+Y \sim \text{Poi}(\lambda_1+\lambda_2)$

$$\boxed{\Pr(X+Y=n) \stackrel{?}{=} e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^n}{n!}} \qquad n = 0, 1, 2, \ldots$$

$$\Pr(X+Y=n) = \sum_{k=0}^{n} \Pr(X=k \cap Y=n-k)$$

$$\stackrel{\text{indep of } X \& Y}{=} \sum_{k=0}^{n} \Pr(X=k)\Pr(Y=n-k) = \sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^k}{k!}\; e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \boxed{\frac{n!}{k!(n-k)!}} \lambda_1^k \lambda_2^{n-k}$$

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k}$$

$$= e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^n}{n!}$$

$\Pr(\ldots k)$ where $W \sim \text{Bin}(n, \frac{\lambda_1}{\ldots})$

$= \Pr(W=k)$ ... $(\lambda_1+\lambda_2)$

$$\boxed{\Pr(X=k \mid X+Y=n)} = \frac{\Pr(X=k \cap X+Y=n)}{\Pr(X+Y=n)}$$

$$= \frac{\Pr(X=k \cap Y=n-k)}{\Pr(X+Y=n)} \underset{\text{independence of } X \,\&\, Y}{=} \frac{\Pr(X=k)\Pr(Y=n-k)}{\Pr(X+Y=n)}$$

$X \sim Poi(\lambda_1)$
$Y \sim Poi(\lambda_2)$
$X+Y \sim Poi(\lambda_1+\lambda_2)$

$$= \frac{\cancel{e^{-\lambda_1}}\frac{\lambda_1^k}{k!} \cdot \cancel{e^{-\lambda_2}}\frac{\lambda_2^{n-k}}{(n-k)!}}{\cancel{e^{-(\lambda_1+\lambda_2)}}\frac{(\lambda_1+\lambda_2)^n}{n!}} = \frac{n!}{k!(n-k)!}\frac{\lambda_1^k\lambda_2^{n-k}}{(\lambda_1+\lambda_2)^n}$$

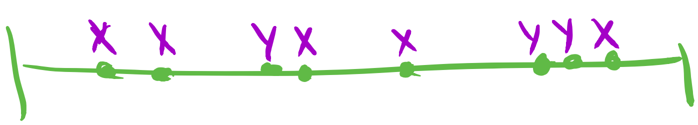
$$= \binom{n}{k}\frac{\lambda_1^k \lambda_2^{n-k}}{(\lambda_1+\lambda_2)^k(\lambda_1+\lambda_2)^{n-k}}$$

$$= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}$$

$Bin(n,p)$

$$= \Pr(W=k)$$

where W is $Bin\left(n, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$



$n=8$

$\Pr(X=k \mid X+Y=n) = \Pr(W=k)$

Conditional expectation of X given event A

$$E(X|A) = \sum_{a \in Range(X)} a \boxed{Pr(X=a|A)}$$

$$\boxed{\text{Law of Total Expectation.}}$$

$$\boxed{E(X) = E(X|A)Pr(A) + E(X|\bar{A})Pr(\bar{A})}$$

Pf

$$E(X) = \sum_{a \in Range(X)} a \underbrace{Pr(X=x)}_{\text{Law of total prob}}$$

E(X|A) ?

$$= \sum_{a \in Range(X)} a \left[Pr(X=x|A) Pr(A) + Pr(X=x|\bar{A}) Pr(\bar{A}) \right]$$

$$\boxed{E(X) = E(X|A)\Pr(A) + E(X|\bar{A})\Pr(\bar{A})}$$

$X \sim Geo(p)$ # tosses to see H's where tosses are indep, each w/ prob p of coming up H.

$$E(X) = \underbrace{E(X \mid \text{first toss is } H)}_{1}\underbrace{\Pr(\text{first toss is } H)}_{p}$$
$$+ \underbrace{E(X \mid \text{first toss is } T)}_{1+E(X)}\underbrace{\Pr(\text{first toss is } T)}_{1-p}$$

T $\underbrace{\underline{T}\ \underline{T}\ \underline{H}\ \_}_{Y}$ # additional tosses to see a H's

$$E(X \mid \text{first } T) = \sum_{k=1}^{\infty}(1+k)\Pr(Y=k)$$
$$= \sum_{k=1}^{\infty}\Pr(Y=k) + \sum_{k=1}^{\infty}k\Pr(Y=k)$$

$$E(X) = p + (1+E(X))(1-p)$$
$$= p + (1-p) + E(X)(1-p)$$
$$= 1 + E(X)(1-p)$$

$$E(X)[1-(1-p)] = 1$$
$$E(X)\,p = 1 \equiv E(X) = \frac{1}{p}$$

$$\mathrm{Var}(X) = E(X^2) - E(X)^2$$

$$E(X^2) = E(X^2 \mid H \text{ on } 1^{st})\, p + E(X^2 \mid \underline{T \text{ on } 1^{st}})(1-p)$$

$$= p + E((1+Y)^2)(1-p)$$

given Tails on 1^{st} toss

$X = 1+Y$

$Y \sim \mathrm{Geo}(p)$

$E(Y) = \frac{1}{p}$

$$= p + E[1 + 2Y + Y^2](1-p)$$

$$= p + \left(1 + \frac{2}{p} + E(Y^2)\right)(1-p)$$

$$= p + (1-p) + 2\frac{(1-p)}{p} + \underbrace{E(Y^2)}_{= E(X^2)}(1-p)$$

$$\Rightarrow \quad E(X^2)\, p = 1 + \frac{2(1-p)}{p}$$

$$E(X^2) = \frac{1}{p} + \frac{2(1-p)}{p^2} = \frac{p + 2(1-p)}{p^2}$$

$$\Rightarrow \mathrm{Var}(X) = \frac{p + 2(1-p)}{p^2} - E(X)^2$$

$$= \frac{p + 2(1-p)}{p^2} - \frac{1}{p^2} = \frac{(1-p)}{p^2}$$