Random variables

A random variable (r.v.) X on a prob space (Ω, P) is a function $X: \Omega \to \mathbb{R}$

$\{X=a\}$ is an event $= \{\omega \mid X(\omega)=a\}$

The expectation or expected value or mean of a r.v. is defined as

$$\boxed{E(X) = \sum_{k \in Range(X)} k \Pr(X=k) = \sum_{\omega \in \Omega} X(\omega) \Pr(\omega)}$$

3 homeworks returned to n students; shuffled randomly

X: # students that receive their own homework

takes values $\{0,1,3\}$

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1 2 3	3
1/6	1 3 2	1
1/6	2 1 3	1
1/6	3 2 1	1
1/6	3 1 2	0
1/6	2 3 1	0

$$E(X) = 0 \cdot \Pr(X=0) + 1 \cdot \Pr(X=1) + 3 \cdot \Pr(X=3)$$

$$= 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{6} = 1$$

$$E(X) = X(123)\Pr(123) + X(132)\Pr(132) + \dots + X(231)\Pr(231)$$

Expectation of a function of a r.v.

$$E(X) = \sum_{k \in Range(X)} k \Pr(X=k) = \sum_{\omega \in \Omega} X(\omega) \Pr(\omega)$$

3 homeworks returned to n students; shuffled randomly

X: # students that receive their own homework

takes values $\{0,1,3\}$

$\Pr(\omega)$	ω	$X(\omega)$	$X^2 \bmod 2$
1/6	1 2 3	3	1
1/6	1 3 2	1	1
1/6	2 1 3	1	1
1/6	3 2 1	1	1
1/6	3 1 2	0	0
1/6	2 3 1	0	0

Example:

$Y = X^2 \bmod 2$

$$E(Y) = 1 \cdot \frac{2}{3} + 0 \cdot \frac{1}{3} = \frac{2}{3}$$

$$E(Y) = f(3)\Pr(X=3) + f(1)\Pr(X=1) + f(0)\Pr(X=0)$$

$$= \underbrace{(3^2 \bmod 2)}_{1} \frac{1}{6} + \underbrace{(1^2 \bmod 2)}_{1} \frac{1}{2} + \underbrace{(0^2 \bmod 2)}_{0} \frac{1}{3}$$

$Y = f(X)$ $\quad f: \mathbb{R} \to \mathbb{R}$

$$E(Y) = \sum_{k \in Range(Y)} k \Pr(Y=k) = \boxed{\sum_{j \in Range(X)} f(j) \Pr(X=j)} = \sum_{\omega \in \Omega} f(X(\omega)) \Pr(\omega)$$

<u>expectation of a function of a r.v.</u>

Linearity of expectation

Thm: $\forall$ 2 random vars X & Y defined on same prob space

$$E(X+Y) = E(X) + E(Y)$$

$Pr(\omega)$	ω	$X(\omega)$	$Y(\omega)$	$(X+Y)(\omega)$
1/6	1 2 3	3	1	4
1/6	1 3 2	1	1	2
1/6	2 1 3	1	1	2
1/6	3 2 1	1	1	2
1/6	3 1 2	0	0	0
1/6	2 3 1	0	0	0

Corollary (by induction)

If $X = X_1 + X_2 + \dots + X_n$

then

$$E(X) = E(X_1) + E(X_2) + \dots + E(X_n)$$

$$E(cX) = cE(X)$$

Proof of Thm: Let $Z = X+Y$

$$E(Z) = \sum_{\omega \in \Omega} Z(\omega) Pr(\omega) = \sum_{\omega \in \Omega} (X(\omega) + Y(\omega)) Pr(\omega)$$

$$= \sum_{\omega \in \Omega} X(\omega) Pr(\omega) + \sum_{\omega \in \Omega} Y(\omega) Pr(\omega) = E(X) + E(Y)$$

Applications of linearity of expectation

① Coin tosses

sequence of n coin tosses, each with prob p of coming up Hs.

$\boxed{X: \# \text{ of heads}}$

$$E(X) = \sum_{k=0}^{n} k \Pr(X=k)$$
$$= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$$

$$\boxed{X_i = \begin{cases} 1 & \text{if } i^{\text{th}} \text{ coin toss is Heads} \\ 0 & \text{o.w.} \end{cases}}$$

$i=2$

HHTT $\quad X_2 = 1$

HTHT $\quad X_2 = 0$

$n=4$

$$E(X_i) = 1 \cdot \Pr(X_i = 1) + 0 \cdot \Pr(X_i = 0)$$
$$= \Pr(X_i = 1) = p$$

$$X = X_1 + X_2 + \dots + X_n$$

by linearity of expectation

$$E(X) = E(X_1) + E(X_2) + \dots + E(X_n)$$
$$= np$$

2 Homeworks

$n=3$

$\Pr(\omega)$	ω	$X(\omega)$	X_1	X_2	X_3
1/6	1 2 3	3	1	1	1
1/6	1 3 2	1	1	0	0
1/6	2 1 3	1	0	0	1
1/6	3 2 1	1	0	1	0
1/6	3 1 2	0	0	0	0
1/6	2 3 1	0	0	0	0

n students HWs returned according to a random perm.

$E(X) = \sum_{k=0}^{n} k \Pr(\text{exactly } k \text{ students get their HW back})$

complicated.

X: # students that get their own homework back

$$X_i = \begin{cases} 1 & i^{th} \text{ person gets their HW back} \\ 0 & \text{o.w.} \end{cases}$$

$X = X_1 + X_2 + \dots + X_n$

$$E(X) = \sum_{i=1}^{n} E(X_i)$$

$$E(X_i) = \Pr(i^{th} \text{ person gets their HW back}) = \frac{(n-1)!}{n!} = \frac{1}{n}$$

$$\Rightarrow \quad E(X) = n \cdot \frac{1}{n} = 1$$

③ Birthdays

m people, each one has random bday $\in \{1, \ldots, 365\}$

X : # pairs of people with the same bday

$$X = \sum_{1 \le i < j \le m} X_{ij}$$

$$X_{ij} = \begin{cases} 1 & \text{if } i \& j \text{ have same bday} \\ 0 & \text{o.w.} \end{cases}$$

$$E(X_{ij}) = \Pr(i \,\&\, j \text{ have same bday}) = \sum_{k=1}^{365} \Pr\left(\begin{matrix} i \,\&\, j \text{ have} \\ \text{bday on} \\ \text{day } k \end{matrix}\right)$$

$$= 365 \left(\frac{1}{365}\right)^2 = \frac{1}{365}$$

$$E(X) = \sum_{1 \le i < j \le m} \underbrace{E(X_{ij})}_{\frac{1}{365}} = \binom{m}{2} \frac{1}{365}$$