Often, several random variables are *simultaneously* observed

X = height and Y = weight

X = cholesterol and Y = blood pressure

X_1, X_2, X_3 = work loads on servers A, B, C

Joint probability mass function:

$f_{XY}(x, y) = P(\{X = x\} \& \{Y = y\})$

Joint cumulative distribution function:

$F_{XY}(x, y) = P(\{X \leq x\} \& \{Y \leq y\})$

Two joint PMFs

W \ Z	1	2	3
1	2/24	2/24	2/24
2	2/24	2/24	2/24
3	2/24	2/24	2/24
4	2/24	2/24	2/24

X \ Y	1	2	3
1	4/24	1/24	1/24
2	0	3/24	3/24
3	0	4/24	2/24
4	4/24	0	2/24

$P(W = Z) = 3 * 2/24 = 6/24$

$P(X = Y) = (4 + 3 + 2)/24 = 9/24$

Can look at arbitrary relationships among variables this way

Two joint PMFs

W \ Z	1	2	3	$f_W(w)$
1	2/24	2/24	2/24	6/24
2	2/24	2/24	2/24	6/24
3	2/24	2/24	2/24	6/24
4	2/24	2/24	2/24	6/24
$f_Z(z)$	8/24	8/24	8/24	

X \ Y	1	2	3	$f_X(x)$
1	4/24	1/24	1/24	6/24
2	0	3/24	3/24	6/24
3	0	4/24	2/24	6/24
4	4/24	0	2/24	6/24
$f_Y(y)$	8/24	8/24	8/24	

Question: Are W & Z independent? Are X & Y independent?

Two joint PMFs

W \ Z	1	2	3	$f_W(w)$
1	2/24	2/24	2/24	6/24
2	2/24	2/24	2/24	6/24
3	2/24	2/24	2/24	6/24
4	2/24	2/24	2/24	6/24
$f_Z(z)$	8/24	8/24	8/24	

X \ Y	1	2	3	$f_X(x)$
1	4/24	1/24	1/24	6/24
2	0	3/24	3/24	6/24
3	0	4/24	2/24	6/24
4	4/24	0	2/24	6/24
$f_Y(y)$	8/24	8/24	8/24	

Marginal PMF of one r.v.: sum over the other (Law of total probability)

$f_Y(y) = \Sigma_x f_{XY}(x,y)$

$f_X(x) = \Sigma_y f_{XY}(x,y)$

Question: Are W & Z independent? Are X & Y independent?

Repeating the Definition: Two random variables X and Y are independent if the events {X=x} and {Y=y} are independent (for any fixed x, y), i.e.

$$\forall x, y \; P(\{X = x\} \,\&\, \{Y=y\}) = P(\{X=x\}) \cdot P(\{Y=y\})$$

Equivalent Definition: Two random variables X and Y are independent if their *joint* probability mass function is the product of their *marginal* distributions, i.e.

$$\forall x, y \; f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

Exercise: Show that this is also true of their *cumulative* distribution functions

A function g(X,Y) defines a new random variable.

Its expectation is:

$$E[g(X, Y)] = \Sigma_x \Sigma_y \, g(x, y) \, f_{XY}(x,y)$$

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Expectation is linear. E.g., if g is linear:

$$E[g(X, Y)] = E[a\,X + b\,Y + c] = a\,E[X] + b\,E[Y] + c$$

Example:

g(X,Y) = 2X-Y

E[g(X,Y)] = 72/24 = 3

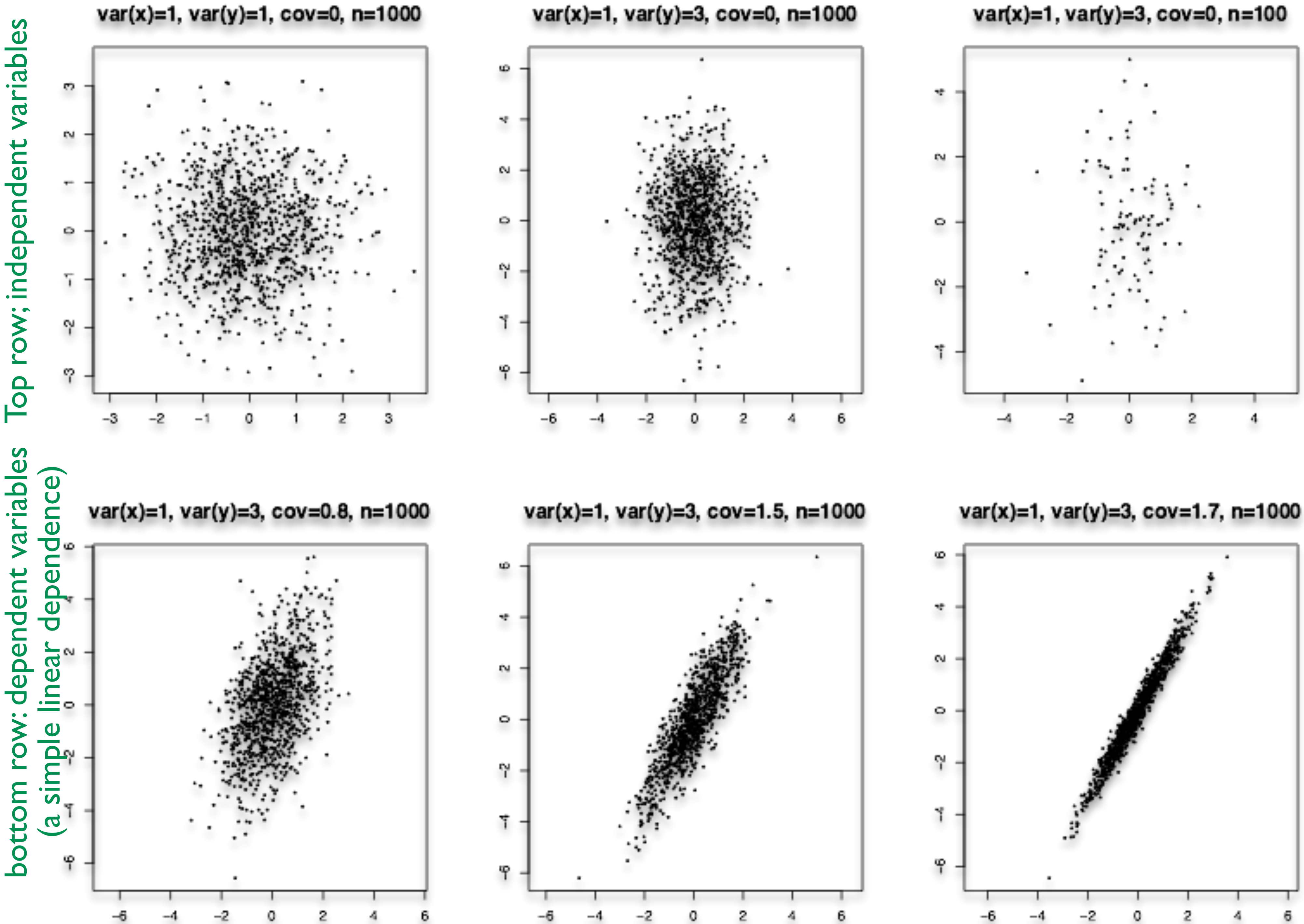
E[g(X,Y)] = 2•E[X] - E[Y]

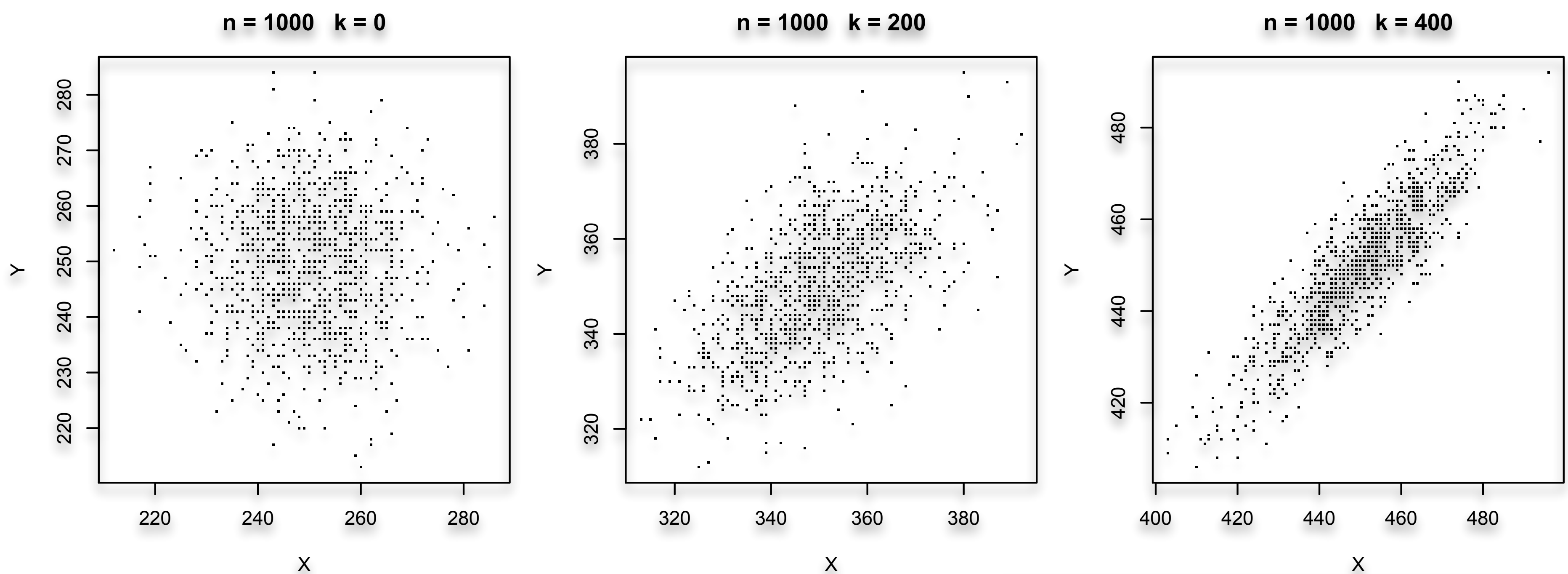
= 2•2.5 - 2 = 3

X \ Y	1	2	3
1	1 • 4/24	0 • 1/24	-1 • 1/24
2	3 • 0/24	2 • 3/24	1 • 3/24
3	5 • 0/24	4 • 4/24	3 • 2/24
4	7 • 4/24	6 • 0/24	5 • 2/24

recall both marginals are uniform

sampling from a joint distribution





Flip n fair coins

X = #Heads seen in first n/2+k

Y = #Heads seen in last n/2+k