Practice with continuous r.v.'s continued.

Law of Total Probability

Continuous

$$Pr(E) = \int_{-\infty}^{\infty} Pr(E \mid Y=y) f_Y(y)\,dy$$

Discrete

$$Pr(E) = \sum_k Pr(E \mid Y=k)\,Pr(Y=k)$$

Law of Total Expectation

Continuous

$$E(X) = \int_{-\infty}^{\infty} E(X \mid Y=y) f_Y(y)\,dy$$

$$\int_{-\infty}^{\infty} x \underbrace{f_{X|Y=y}(x)}_{\frac{f_{XY}(x,y)}{f_Y(y)}}\,dx$$

Discrete

$$E(X) = \sum_k \underbrace{E(X \mid Y=k)}_{\sum_x x\, \underbrace{Pr(X=x \mid Y=k)}_{\frac{Pr(X=x \cap Y=k)}{Pr(Y=k)}}} Pr(Y=k)$$

Joint Distributions

Joint CDF: $F_{XY}(x,y) = Pr(X \le x, Y \le y)$

$$F(a,b) = \int_{-\infty}^{a} \int_{-\infty}^{b} \underbrace{f_{XY}(x,y)}_{\text{joint density fn.}} \, dy \, dx$$

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \, dx \, dy = 1$$

$$Pr(a < X < a+da, \ b < Y < b+db) = \int_{a}^{a+da} \int_{b}^{b+db} f(x,y) \, dy \, dx \approx f(a,b) \, da \, db$$

Marginal density fn

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y) \, dy$$

Independence & Conditional prob mass fns

X & Y indep $\equiv$ $f_{XY}(x,y) = f_X(x) f_Y(y)$ $\forall x, y$

for y with $f_Y(y) > 0$

$$\underbrace{f_{X|Y=y}(x) = \frac{f_{XY}(x,y)}{f_Y(y)}}_{\text{conditional prob density fn}}$$

$$Pr(X=x \mid Y=y) = \frac{Pr(X=x, Y=y)}{Pr(Y=y)}$$

$$f_{X,Y}(x,y) = f_Y(y) f_{X|Y=y}(x) = f_X(x) f_{Y|X=x}(y)$$

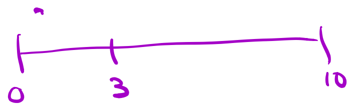
Example

$$E(Y) = \sum_{k=0}^{\infty} E(Y \mid X=k)\Pr(X=k)$$

$X \sim \text{Unif}(0,10)$

$f_X(x) = \begin{cases} \frac{1}{10} & 0 \le x \le 10 \\ 0 & \text{o.w} \end{cases}$

$Y \sim \text{Unif}[0, X]$



if $X=3$

$Y_{|X=3} \sim \text{Unif}[0,3]$

$f_{Y|X=3}(y) = \begin{cases} \frac{1}{3} & 0 \le y \le 3 \\ 0 & \text{o.w.} \end{cases}$

$$E(Y) = \int_0^{10} E(Y \mid X=x) f_X(x)\,dx$$

law of total expectation

$$= \int_0^{10} \frac{x}{2} \cdot \frac{1}{10}\,dx = 2.5$$

$Y \mid X=x \sim \text{Unif}(0,x)$

$$f_{XY}(x,y) = \underbrace{f_X(x)}_{\text{Unif}(0,10)} \underbrace{f_{Y|X=x}(y)}_{\text{Unif}(0,x)}$$

discrete setting
(X & Y discrete r.v.'s)

$\Pr(X=x, Y=y)$

$= \Pr(X=x \mid Y=y)\Pr(Y=y)$

$= \Pr(Y=y \mid X=x)\Pr(X=x)$

$$= \begin{cases} \frac{1}{10} \cdot \frac{1}{x} & 0 < y \le x < 10 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_y^{10} \frac{1}{10x}\,dx \\ = \frac{1}{10} \ln x \Big|_y^{10} = \frac{1}{10}\ln\left(\frac{10}{y}\right) & 0 < y < 10 \\ 0 & \text{o.w.} \end{cases}$$

not independent.

$$f(x,y) = \begin{cases} x e^{-x(y+1)} & x>0,\ y>0 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{X|Y=y} = \frac{f(x,y)}{f_Y(y)}$$

$$\boxed{f_Y(y) = \int_0^\infty x e^{-x(y+1)}\, dx}$$

Find density of $Z = XY$

$$F_Z(z) = Pr(Z \le z) = Pr(XY \le z)$$

$$= \int_0^\infty Pr(XY \le z \mid X = x)\, f_X(x)\, dx$$

$$= \int_0^\infty Pr(xY \le z)\, f_X(x)\, dx$$

$$= \int_0^\infty \underbrace{Pr\left(Y \le \tfrac{z}{x}\right)}_{F_Y\left(\frac{z}{x}\right)} f_X(x)\, dx$$

$$E(W) = \frac{1}{\lambda}$$

$$W \sim exp(\lambda)$$

$$\frac{1}{\lambda} = E(W) = \int_0^\infty x\lambda e^{-\lambda x}\, dx$$

$$F_Y(y) = \int_{-\infty}^{y} f_Y(z)\, dz$$

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$