

Practice with continuous r.v.'s

Law of Total Probability

Continuous

$$Pr(E) = \int_{-\infty}^{\infty} Pr(E|Y=y) f_Y(y) dy$$

Discrete

$$Pr(E) = \sum_k Pr(E|Y=k) Pr(Y=k)$$

Law of Total Expectation

Continuous

$$E(X) = \int_{-\infty}^{\infty} E(X|Y=y) f_Y(y) dy$$



$$\int_{-\infty}^{\infty} x \underbrace{f_{X|Y=y}(x)}_{\frac{f_{XY}(x,y)}{f_Y(y)}} dx$$

Discrete

$$E(X) = \sum_k \underbrace{E(X|Y=k)}_{\frac{\sum_x Pr(X=x|Y=k)}{Pr(Y=k)}} Pr(Y=k)$$

Example

accidents a ^{random} person has in a year
is Poisson (λ) but λ is r.v.

$$\lambda \sim \text{exp}(1)$$

fraction of population that has $\lambda < x$
is $1 - e^{-x} = F_{\text{exp}(1)}(x)$

$$\begin{aligned} & \Pr(\text{random person has no accidents}) \\ &= \int_0^{\infty} \Pr(\text{random person has no acc} \mid \lambda=y) \underbrace{f_X(y)}_{\text{X} \sim \text{exp}(1)} dy \end{aligned}$$

law of
total prob

$$\begin{aligned} &= \int_0^{\infty} e^{-y} e^{-y} dy \\ &= \int_0^{\infty} e^{-2y} dy = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} & E(\text{\# accidents for random person in year}) \\ &= \int_0^{\infty} \underbrace{E(\text{\#acc} \mid \lambda=y)}_{\text{Poisson}(\lambda=y)} f_Y(y) dy \\ &= \int_0^{\infty} y e^{-y} dy = 1 \end{aligned}$$

law of
total exp

instad $\lambda \sim \text{Bin}(100, \frac{1}{5})$

$$\sum_{k=0}^{100} \Pr(\text{no accid} \mid \lambda=k) \Pr(\lambda=k)$$

$$\binom{100}{k} \left(\frac{1}{5}\right)^k \left(\frac{4}{5}\right)^{100-k}$$

$$\begin{aligned} & \text{exp}(\lambda) \\ & f_X(x) = \lambda e^{-\lambda x} \end{aligned}$$

$$\begin{aligned} & X \sim \text{Poisson}(\lambda) \\ & \Pr(X=k) = e^{-\lambda} \frac{\lambda^k}{k!} \end{aligned}$$

$$E(X) = \sum_{k=0}^{\infty} E(X \mid \lambda=k) \cdot \Pr(\lambda=k)$$

law of total prob:

$$\begin{aligned} & \Pr(X < Y) \\ &= \int_{-\infty}^{\infty} \underbrace{\Pr(X < Y \mid Y=y)}_{\substack{\Pr(X < y) \\ F_X(y)}} f_Y(y) dy \end{aligned}$$

X r.v. $Y = g(X)$
 prob density fn for Y

Recipe for computing $f_Y(y)$

① Find CDF for Y

$$F_Y(y) = \Pr(Y \leq y) = \Pr(g(X) \leq y) = \Pr(X \leq g^{-1}(y))$$

$$\left[= \int_{x|g(x) \leq y} f_X(x) dx \right] \quad \text{only if } g \text{ invertible}$$

② differentiate to get p.d.f.

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

	X	X ²
0	1	1
1	2	4
2	3	9
3	4	16
4	5	25

find density of $Y = 2X$
 X $f_X(x)$ $F_X(x)$

$$\begin{aligned} \textcircled{1} F_Y(y) &= \Pr(Y \leq y) \\ &= \Pr(2X \leq y) \\ &= \Pr(X \leq \frac{y}{2}) \\ &= F_X(\frac{y}{2}) \end{aligned}$$

$$\begin{aligned} \textcircled{2} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy} F_X(\frac{y}{2}) \\ &= f_X(\frac{y}{2}) \cdot \frac{1}{2} \end{aligned}$$

$$\begin{aligned} W &= X^2 \\ F_W(w) &= \Pr(X^2 \leq w) \\ &= \Pr(-\sqrt{w} \leq X \leq \sqrt{w}) \end{aligned}$$

intuition, i.e. to see this makes sense

$$\Pr(a - \frac{\epsilon}{2} \leq X \leq a + \frac{\epsilon}{2}) \approx \epsilon f_X(a)$$

for epsilon small

$$Y = 2X$$

$$\begin{aligned} \epsilon \cdot f_Y(a) &\approx \Pr(a - \frac{\epsilon}{2} \leq Y \leq a + \frac{\epsilon}{2}) \\ &= \Pr(a - \frac{\epsilon}{2} \leq 2X \leq a + \frac{\epsilon}{2}) \\ &= \Pr(\frac{a}{2} - \frac{\epsilon}{4} \leq X \leq \frac{a}{2} + \frac{\epsilon}{4}) \\ &\approx \frac{\epsilon}{2} \cdot f_X(\frac{a}{2}) \\ &= \epsilon \cdot \frac{1}{2} f_X(\frac{a}{2}) \end{aligned}$$

$$X \sim N(\mu, \sigma^2) \quad Y = aX + b$$

$$\text{Claim: } Y \sim N(a\mu + b, a^2\sigma^2)$$

Proof $F_Y(y) = \Pr(Y \leq y) = \Pr(aX + b \leq y)$

$$= \Pr\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

$$\frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{a} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{\left(\frac{y-b}{a} - \mu\right)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi} a\sigma} e^{-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}}$$

which is density of $N(a\mu + b, a^2\sigma^2)$

Joint Distributions

Joint CDF: $F_{XY}(x,y) = \Pr(X \leq x, Y \leq y)$

$$F(a,b) = \int_{-\infty}^a \int_{-\infty}^b \underbrace{f_{XY}(x,y)}_{\text{joint density fn.}} dy dx$$

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = 1$$

$$\Pr(a < X < a+da, b < Y < b+db) = \int_a^{a+da} \int_b^{b+db} f(x,y) dy dx \approx f(a,b) da db$$

Marginal density fn

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y) dy$$

Independence & Conditional prob mass fns

$$X \& Y \text{ indep} \equiv f_{XY}(x,y) = f_X(x) f_Y(y) \quad \forall x,y$$

$$\text{for } y \text{ with } f_Y(y) > 0 \quad \underbrace{f_{X|Y=y}(x) = \frac{f_{XY}(x,y)}{f_Y(y)}}_{\text{conditional prob density fn}}$$

$$f_{X,Y}(x,y) = f_Y(y) f_{X|Y=y}(x) = f_X(x) f_{Y|X=x}(y)$$

Example

$$X \sim \text{Unif}(0, 10)$$

$$f_X(x) = \begin{cases} \frac{1}{10} & 0 \leq x \leq 10 \\ 0 & \text{o.w.} \end{cases}$$

$$Y \sim \text{Unif}[0, X]$$



if $X=3$

$$Y|_{X=3} \sim \text{Unif}[0, 3]$$

$$f_{Y|X=3}(y) = \begin{cases} \frac{1}{3} & 0 \leq y \leq 3 \\ 0 & \text{o.w.} \end{cases}$$

$$E(Y) = \int_0^{10} \underbrace{E(Y|X=x)}_{\frac{x}{2}} f_X(x) dx$$

$$= \int_0^{10} \frac{x}{2} \cdot \frac{1}{10} dx = 2.5$$

law of total expectation

$$f_{XY}(x, y) = \underbrace{f_X(x)}_{\text{Unif}(0, 10)} \underbrace{f_{Y|X=x}(y)}_{\text{Unif}(0, x)}$$

$$= \frac{1}{10} \cdot \frac{1}{x}$$

discrete setting
(X & Y discrete r.v.'s)

$$\Pr(X=x, Y=y)$$

$$= \Pr(X=x | Y=y) \Pr(Y=y)$$

$$= \Pr(Y=y | X=x) \Pr(X=x)$$