CSE 312: Foundations of Computing II

Section 7: Joint Distributions Solutions

0. Review of Main Concepts

(a) **Joint Probability Mass Function**: Discrete random variables X and Y have the joint probability mass function

$$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y).$$

Their joint range or support is denoted $\Omega_{X,Y} = \{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$. Note that $\sum_{(x,y)\in\Omega_{X,Y}} p_{X,Y}(x,y) = 1$. If Z were also another discrete random variable, then X, Y, Z would have joint probability mass function $p_{X,Y,Z}(x,y,z) = \mathbb{P}(X = x, Y = y, Z = z)$.

(b) **Marginal Probability Mass Function**: Suppose X, Y are jointly distributed discrete random variables. Then, the marginal pmf of X is

$$p_X(x) = \sum_{y\in\Omega_Y} p_{X,Y}(x,y).$$

(c) **Expectation**: Suppose X, Y are discrete random variables and g is a function. Then,

$$\mathbb{E}[g(X,Y)] = \sum_{(x,y)\in\Omega_{X,Y}} g(x,y)p_{X,Y}(x,y).$$

(d) **Joint Cumulative Distribution Function**: X and Y have joint CDF $F_{X,Y}(x,y) = \mathbb{P}(X \le x, Y \le y)$. If X, Y are discrete, we can compute this as

$$F_{X,Y}(x,y) = \sum_{s\le x, t\le y} p_{X,Y}(s,t).$$

(e) **Independence**: X and Y are independent discrete random variables if and only if

$$p_{X,Y}(x,y) = p_X(x)p_Y(y) \quad \forall (x,y) \in \Omega_X \times \Omega_Y.$$

Note that a necessary but not sufficient condition for independence is that $\Omega_{X,Y} = \Omega_X \times \Omega_Y$.

1. Joint PMF's

Suppose X and Y have the following joint PMF:

X/Y	1	2	3
0	0	0.2	0.1
1	0.3	0	0.4

(a) Identify the range of X (Ω_X), the range of Y (Ω_Y), and their joint range ($\Omega_{X,Y}$).

Solution:

$\Omega_X = \{0,1\}$, $\Omega_Y = \{1,2,3\}$, and $\Omega_{X,Y} = \{(0,2),(0,3),(1,1),(1,3)\}$

(b) Find the marginal PMF for X, $p_X(x)$ for $x \in \Omega_X$.

Solution:

$$p_X(0) = \sum_y p_{X,Y}(0,y) = 0 + 0.2 + 0.1 = 0.3$$
$$p_X(1) = 1 - p_X(0) = 0.7$$

(c) Find the marginal PMF for Y, $p_Y(y)$ for $y \in \Omega_Y$.

Solution:

$$p_Y(1) = \sum_x p_{X,Y}(x,1) = 0 + 0.3 = 0.3$$
$$p_Y(2) = \sum_x p_{X,Y}(x,2) = 0.2 + 0 = 0.2$$
$$p_Y(3) = \sum_x p_{X,Y}(x,3) = 0.1 + 0.4 = 0.5$$

(d) Are X and Y independent? Why or why not?

Solution:

No, since a necessary condition is that $\Omega_{X,Y} = \Omega_X \times \Omega_Y$.

(e) Find $\mathbb{E}\left[X^3Y\right]$.

Solution:

Note that $X^3 = X$ since X takes values in $\{0,1\}$.

$$\mathbb{E}\left[X^3Y\right] = \mathbb{E}[XY] = \sum_{(x,y)\in\Omega_{X,Y}} xyp_{X,Y}(x,y) = 1\cdot 1\cdot 0.3 + 1\cdot 3\cdot 0.4 = 1.5$$

2. Trinomial Distribution

A generalization of the binomial model is when there is a sequence of n independent trials, but with three outcomes, where $\mathbb{P}(\text{outcome } i) = p_i$ for $i = 1,2,3$ and of course $p_1 + p_2 + p_3 = 1$. Let X_i be the number of times outcome i occurred for $i = 1,2,3$, where $X_1 + X_2 + X_3 = n$. Find the joint PMF $p_{X_1,X_2,X_3}(x_1,x_2,x_3)$ and specify its value for all $x_1, x_2, x_3 \in \mathbb{R}$.

Solution:

Same argument as for the binomial PMF:

$$p_{X_1,X_2,X_3}(x_1,x_2,x_3) = \binom{n}{x_1,x_2,x_3}\prod_{i=1}^{3} p_i^{x_i} = \frac{n!}{x_1!x_2!x_3!}p_1^{x_1}p_2^{x_2}p_3^{x_3}$$

where $x_1 + x_2 + x_3 = n$ and are nonnegative integers.

3. More Die Rolling...

Suppose we roll a fair four-sided die independently twice, and let X be the value on the first roll, and Y the value on the second roll.

(a) Let $S = \max\{X,Y\}$ and $T = \min\{X,Y\}$. Find the joint range $\Omega_{S,T}$ and the joint PMF $p_{S,T}(s,t)$. Are S and T independent?

Solution:

The joint range is any ordered pair where the below PMF is nonzero.

S/T	1	2	3	4
1	1/16	0	0	0
2	2/16	1/16	0	0
3	2/16	2/16	1/16	0
4	2/16	2/16	2/16	1/16

They are not independent since $\Omega_{S,T} \neq \Omega_S \times \Omega_T$.

(b) Find $p_{S|T}(s|3)$.

Solution:

$$p_T(3) = \sum_s p_{S,T}(s,3) = 1/16 + 2/16 = 3/16$$

$$p_{S|T}(s|3) = \frac{p_{S,T}(s,3)}{p_T(3)}$$

$$p_{S|T}(1|3) = p_{S|T}(2|3) = 0$$

$$p_{S|T}(3|3) = \frac{p_{S,T}(3,3)}{p_T(3)} = \frac{1/16}{3/16} = 1/3$$

$$p_{S|T}(4|3) = \frac{p_{S,T}(4,3)}{p_T(3)} = \frac{2/16}{3/16} = 2/3$$

(c) Find $\mathbb{E}[ST]$.

Solution:

Note S and T are not independent, so we cannot say that $\mathbb{E}[ST] = \mathbb{E}[S]\,\mathbb{E}[T]$. However, make the clever observation that $ST = XY$ always. Since $X, Y \sim$ Uniform$(1, 4)$, we have $\mathbb{E}[X] = \mathbb{E}[Y] = 2.5$. By independence of X and Y, $\mathbb{E}[ST] = \mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y] = 2.5^2 = 6.25$.