CSE 312: Foundations of Computing II

Conditional Expectation 6 Solutions

Review of Main Concepts

(a) **Law of Total Expectation (Event Version)**: Let X be a random variable, and let events $A_1, ..., A_n$ partition the sample space. Then,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i] \, \mathbb{P}(A_i)$$

(b) **Conditional Expectation**: Let X and Y be discrete random variables. Then, the conditional expectation of X given $Y = y$ is

$$\mathbb{E}[X \mid Y = y] = \sum_x x p_{X|Y}(x|y) = \sum_x x \mathbb{P}(X = x | Y = y)$$

Linearity of expectation still applies to conditional expectation: $\mathbb{E}[X + Y \mid A] = \mathbb{E}[X \mid A] + \mathbb{E}[Y \mid A]$

(c) **Law of Total Expectation (RV Version)**: Suppose X and Y be discrete random variables. Then,

$$\mathbb{E}[X] = \sum_y \mathbb{E}[X \mid Y = y] \, p_Y(y)$$

Trapped Miner

A miner is trapped in a mine containing 3 doors.

- D_1: The 1^{st} door leads to a tunnel that will take him to safety after 3 hours.
- D_2: The 2^{nd} door leads to a tunnel that returns him to the mine after 5 hours.
- D_3: The 3^{rd} door leads to a tunnel that returns him to the mine after a number of hours that is Binomial with parameters $(12, \frac{1}{3})$.

At all times, he is equally likely to choose any one of the doors. What is the expected number of hours for this miner to reach safety?

Solution:

Let T = number of hours for the miner to reach safety. (T is a random variable)
Let D_i be the event the i^{th} door is chosen. $i \in \{1, 2, 3\}$. By Law of Total Expectation:

$$\begin{aligned}
\mathbb{E}[T] &= \mathbb{E}[T \mid D_1] \, \mathbb{P}(D_1) + \mathbb{E}[T \mid D_2] \, \mathbb{P}(D_2) + \mathbb{E}[T \mid D_3] \, \mathbb{P}(D_3) \\
&= 3 \cdot \frac{1}{3} + (5 + \mathbb{E}[T]) \cdot \frac{1}{3} + (4 + \mathbb{E}[T]) \cdot \frac{1}{3} \\
\mathbb{E}[T] &= 12
\end{aligned}$$

Therefore, the expected number of hours for this miner to reach safety is 12.

Elevator

The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where the others get off, compute the expected number of stops that the elevator will make before discharging all the passengers. Assume an infinitely large elevator.

Solution:

Let X = number of people who enter the elevator. $X \sim Poi(10)$. Let Y = number of stops. (X, Y are both random variables)

$$\mathbb{E}[Y] = \sum_{k=0}^{\infty} E[Y|X = k]\mathbb{P}(X = k)$$
$$\mathbb{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!} = e^{-10}\frac{10^k}{k!}$$

Let Y_i be an indicator random variable. $Y_i =$ whether the elevator stops at the i^{th} floor.

$$\mathbb{E}[Y \mid X = k] = E[Y_1 + Y_2 + ... + Y_N|X = k]$$
$$\mathbb{E}[Y_i \mid X = k] = 1 - (\frac{N-1}{N})^k$$

By linearity of expectation:

$$\mathbb{E}[Y \mid X = k] = \sum_{i=1}^{N} \mathbb{E}[Y_i \mid X = k] = N \cdot (1 - (\frac{N-1}{N})^k)$$

Finally, we put everything together:

$$\mathbb{E}[Y] = \sum_{k=0}^{\infty} (N \cdot (1 - (\frac{N-1}{N})^k)) \cdot (e^{-10}\frac{10^k}{k!})$$

Lemonade Stand

Suppose I run a lemonade stand, which costs me $100 a day to operate. I sell a drink of lemonade for $20. Every person who walks by my stand either buys a drink or doesn't (no one buys more than one). If it is raining, n_1 people walk by my stand, and each buys a drink independently with probability p_1. If it isn't raining, n_2 people walk by my stand, and each buys a drink independently with probability p_2. It rains each day with probability p_3, independently of every other day. Let X be my profit over the next week. What is $\mathbb{E}[X]$?

Solution:

Let R be the event it rains. Let X_i be how many drinks I sell on day i for $i = 1, ..., 7$. We are interested in $X = \sum_{i=1}^{7}(20X_i - 100)$. We have $X_i|R \sim$ Binomial(n_1, p_1), so $\mathbb{E}[X_i \mid R] = n_1p_1$. Similarly, $X_i|R^C \sim$ Binomial(n_2, p_2), so $\mathbb{E}\left[X_i \mid R^C\right] = n_2p_2$. By the law of total expectation,

$$\mu = \mathbb{E}[X_i] = \mathbb{E}[X_i \mid R]\,\mathbb{P}(R) + \mathbb{E}\left[X_i \mid R^C\right]\mathbb{P}(R^C) = n_1p_1p_3 + n_2p_2(1 - p_3)$$

Hence, by linearity of expectation,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{7}(20X_i - 100)\right] = 20\sum_{i=1}^{7}\mathbb{E}[X_i] - 700 = 140\mu - 700$$

Particle Emissions

Suppose we are measuring particle emissions, and the number of particles emitted follows a Poisson distribution with parameter λ, $X \sim \text{Poisson}(\lambda)$. Suppose our device to measure emissions is not always entirely accurate sometimes we fail to observe particles that actually are admitted. So for each particle actually emitted, say we have probability p close to 1 of actually recording it, independently of other particles. Let Y be the number of particles we observed. What distribution does Y follow with what parameters, and what is $\mathbb{E}[Y]$?

Solution:

$$\begin{aligned}
p_Y(y) &= \mathbb{P}(Y = y) \\
&= \sum_{x=y}^{\infty} \mathbb{P}(Y = y | X = x)\mathbb{P}(X = x) \quad \text{(Law of Total Expectation)} \\
&= \sum_{x=y}^{\infty} \binom{x}{y} p^y (1-p)^{x-y} \cdot e^{-\lambda} \frac{\lambda^x}{x!} \quad \text{(Plug in Poisson and Binomial PMFs)} \\
&= e^{-\lambda} p^y \sum_{x=y}^{\infty} \frac{x!}{y!(x-y)!} (1-p)^{x-y} \frac{\lambda^x}{x!} \\
&= \frac{e^{-\lambda} p^y}{y!} \sum_{x=y}^{\infty} \frac{\lambda^x}{(x-y)!} (1-p)^{x-y} \\
&= \frac{e^{-\lambda} p^y}{y!} \sum_{k=0}^{\infty} \frac{\lambda^{k+y}}{k!} (1-p)^k \quad (\text{let } k = x - y) \\
&= \frac{e^{-\lambda} (\lambda p)^y}{y!} \sum_{k=0}^{\infty} \frac{(\lambda(1-p))^k}{k!} \\
&= \frac{e^{-\lambda} (\lambda p)^y}{y!} \cdot e^{-\lambda(1-p)} \quad (\text{Taylor series for } e^{-\lambda(1-p)}) \\
&= \frac{e^{-p\lambda} (\lambda p)^y}{y!}
\end{aligned} \tag{1}$$

So $Y \sim \text{Poisson}(p\lambda)$ and $\mathbb{E}[Y] = p\lambda$.