CSE 312: Foundations of Computing II

Random Variables, Linearity of Expectation 4 Solutions

Review of Main Concepts

(a) **Random Variable (rv)**: A numeric function $X : \Omega \to \mathbb{R}$ of the outcome.

(b) **Range/Support:** The support/range of a random variable X, denoted Ω_X, is the set of all possible values that X can take on.

(c) **Discrete Random Variable (drv)**: A random variable taking on a countable (either finite or countably infinite) number of possible values.

(d) **Probability Mass Function (pmf) for a discrete random variable $\mathbf{X}$**: a function $p_X : \Omega_X \to [0, 1]$ with $p_X(x) = \mathbb{P}(X = x)$ that maps possible values of a discrete random variable to the probability of that value happening, such that $\sum_x p_X(x) = 1$.

(e) **Expectation (expected value, mean, or average)**: The expectation of a discrete random variable is defined to be $\mathbb{E}[X] = \sum_x x p_X(x) = \sum_x x\mathbb{P}(X = x)$. The expectation of a function of a discrete random variable $g(X)$ is $\mathbb{E}[g(X)] = \sum_x g(x)p_X(x)$.

(f) **Linearity of Expectation**: Let X and Y be random variables, and $a, b, c \in \mathbb{R}$. Then, $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$.

Identify that Range!

Identify the support/range Ω_X of the random variable X, if X is...

(a) The sum of two rolls of a six-sided die.

Solution:

$\Omega_X = \{2, 3, ..., 12\}$

(b) The number of lottery tickets I buy until I win it.

Solution:

$\Omega_X = \{1, 2, ...\} = \mathbb{N}$

(c) The number of heads in n flips of a coin with $0 < \mathbb{P}(\text{head}) < 1$.

Solution:

$\Omega_X = \{0, 1, ..., n\}$

(d) The number of heads in n flips of a coin with $\mathbb{P}(\text{head}) = 1$.

Solution:

$\Omega_X = \{n\}$

(e) The time I wait at the bus stop for the next bus.

Solution:

$\Omega_X = [0, \infty)$

(f) The number of people born in the next year.

Solution:

$\Omega_X = \{0, 1, 2, ...\}$

Coin Flipping

Suppose we have a coin with probability p of heads. Suppose we flip this coin until we flip a head for the first time. Let X be the number of times we flip the coin *up to and including* the first head. What is $\mathbb{P}(X = k)$, for $k = 1, 2, \ldots$? Verify that $\sum_{k=1}^{\infty} \mathbb{P}(X = k) = 1$, as it should. (You may use the fact that $\sum_{j=0}^{\infty} a^j = \frac{1}{1-a}$ for $|a| < 1$).

Solution:

$$\mathbb{P}(X = k) = (1-p)^{k-1} p$$

If the k^{th} flip is our first head, the first $k-1$ must be tails. Then the k^{th} flip must be a head.

$$\sum_{k=1}^{\infty} \mathbb{P}(X = k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p = p \sum_{j=0}^{\infty} (1-p)^j = \frac{p}{1-(1-p)} = 1$$

Kit Kats Again

Suppose we have N candies in a jar, K of which are kit kats. Suppose we draw (without replacement) until we have (exactly) k kit kats, $k \leq K \leq N$. Let X be the number of draws until the k^{th} kit kat. What is Ω_X, the range of X? What is $p_X(n) = \mathbb{P}(X = n)$? (We say X is a "negative hypergeometric" random variable).

Solution:

$$\Omega_X = \{k, k+1, \ldots N - K + k\}$$

$$p_X(n) = \mathbb{P}(X = n) = \frac{\binom{K}{k-1}\binom{N-K}{n-k}}{\binom{N}{n-1}} \, \frac{K-(k-1)}{N-(n-1)}, \; n \in \Omega_X$$

Hungry Washing Machine

You have 10 pairs of socks (so 20 socks in total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks chosen at random. Every subset of 4 socks is equally probable to be the subset that gets eaten. Let X be the number of complete pairs of socks that you have left.

(a) What is the range of X, Ω_X (the set of possible values it can take on)? What is the probability mass function of X?

Solution:

$$\Omega_X = \{6, 7, 8\}$$

$$p_X(k) = \begin{cases} \frac{\binom{10}{4}2^4}{\binom{20}{4}} & k = 6 \\ \frac{10\binom{9}{2}2^2}{\binom{20}{4}} & k = 7 \\ \frac{\binom{10}{2}}{\binom{20}{4}} & k = 8 \end{cases}$$

(b) Find $\mathbb{E}[X]$ from the definition of expectation.

Solution:

$$\mathbb{E}[X] = 6 \cdot \frac{\binom{10}{4}2^4}{\binom{20}{4}} + 7 \cdot \frac{10\binom{9}{2}2^2}{\binom{20}{4}} + 8 \cdot \frac{\binom{10}{2}}{\binom{20}{4}} = \frac{120}{19}$$

(c) Find $\mathbb{E}[X]$ using linearity of expectation.

Solution:

For $i \in [10]$, let X_i be 1 if pair i survived, and 0 otherwise. Then, $X = \sum_{i=1}^{10} X_i$. But $\mathbb{E}[X_i] = 1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = \frac{\binom{18}{4}}{\binom{20}{4}}$. Hence,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{10} X_i\right] = \sum_{i=1}^{10} \mathbb{E}[X_i] = \sum_{i=1}^{10} \frac{\binom{18}{4}}{\binom{20}{4}} = 10\frac{\binom{18}{4}}{\binom{20}{4}} = \frac{120}{19}$$

(d) Which way was easier? Doing both (a) and (b), or just (c)?

Solution:

Part (c).

Hat Check

At a reception, n people give their hats to a hat-check person. When they leave, the hat-check person gives each of them a hat chosen at random from the hats that remain. What is the expected number of people who get their own hats back? (Notice that the hats returned to two people are not independent events: if a certain hat is returned to one person, it cannot also be returned to the other person.)

Solution:

Let X be the number of people who get their hats back. For $i \in [n]$, let X_i be 1 if person i gets their hat back, and 0 otherwise. Then, $\mathbb{E}[X_i] = \mathbb{P}(X_i = 1) = \frac{1}{n}$, and $X = \sum_{i=1}^{n} X_i$. Hence

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \frac{1}{n} = n \cdot \frac{1}{n} = 1$$

For this section, we expect to end here (or before!). The rest of these problems can be done at home for extra practice, or if you finish 1-5 early. Solutions will be posted.

More Coin Flipping ...

Suppose we have a coin with probability p of heads. Suppose we flip this coin n times independently. Let X be the number of heads that we observe. What is $\mathbb{P}(X = k)$, for $k = 0, \dots n$? Verify that $\sum_{k=0}^{n} \mathbb{P}(X = k) = 1$, as it should.

Solution:

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

For a given sequence with exactly k heads, the probability of that sequence is $p^k(1-p)^{n-k}$. However, there are $\binom{n}{k}$ such sequences, so the probability of exactly k heads is $\binom{n}{k} p^k (1-p)^{n-k}$.

$$\sum_{k=0}^{n} \mathbb{P}(X = k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1$$

The middle equality uses the Binomial Theorem.

Frogger

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability p_1, to the left with probability p_2, and doesn't move with probability p_3, where $p_1 + p_2 + p_3 = 1$. After 2 seconds, let X be the location of the frog.

(a) Find $p_X(k)$, the probability mass function for X.

Solution:

Let L be a left step, R be a right step, and N be no step.

The range of X is $\{-2, -1, 0, 1, 2\}$. We can compute $p_X(-2) = \mathbb{P}(X = -2) = \mathbb{P}(LL) = p_2^2$, $p_X(-1) = \mathbb{P}(X = -1) = \mathbb{P}(LN \cup NL) = 2p_2p_3$, and $p_X(0) = \mathbb{P}(X = 0) = \mathbb{P}(NN \cup LR \cup RL) = p_3^2 + 2p_1p_2$. Similarly for $p_X(1)$ and $p_X(2)$.

$$p_X(k) = \begin{cases} p_2^2 & k = -2 \\ 2p_2p_3 & k = -1 \\ p_3^2 + 2p_1p_2 & k = 0 \\ 2p_1p_3 & k = 1 \\ p_1^2 & k = 2 \end{cases}$$

(b) Compute $\mathbb{E}[X]$ from the definition.

Solution:

$$\mathbb{E}[X] = (-2)(p_2^2) + (-1)(2p_2p_3) + (0)(p_3^2 + 2p_1p_2) + (1)(2p_1p_3) + (2)(p_1^2) = 2(p_1 - p_2)$$

(c) Compute $\mathbb{E}[X]$ again, but using linearity of expectation.

Solution:

Let Y be the amount you moved on the first step (either $-1, 0, 1$), and Z the amount you moved on the second step. Then, $\mathbb{E}[Y] = \mathbb{E}[Z] = (1)(p_1) + (0)(p_3) + (-1)(p_2) = p_1 - p_2$.

Then $X = Y + Z$ and $\mathbb{E}[X] = \mathbb{E}[Y + Z] = \mathbb{E}[Y] + \mathbb{E}[Z] = 2(p_1 - p_2)$

Balls in Bins

Let X be the number of bins that remain empty when m balls are distributed into n bins randomly and independently. For each ball, each bin has an equal probability of being chosen. (Notice that two bins being empty are not independent events: if one bin is empty, that decreases the probability that the second bin will also be empty. This is particularly obvious when $n = 2$ and $m > 0$.) Find $\mathbb{E}[X]$.

Solution:

For $i \in [n]$, let X_i be 1 if bin i is empty, and 0 otherwise. Then, $X = \sum_{i=1}^{n} X_i$. We first compute $\mathbb{E}[X_i] = 1 \cdot \mathbb{P}(X_i = 1) + 0 \cdot \mathbb{P}(X_i = 0) = \mathbb{P}(X_i = 1) = (\frac{n-1}{n})^m$. Hence,

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = n \cdot (\frac{n-1}{n})^m$$