CSE 312: Foundations of Computing II

Section 3: Conditional Probability Solutions

0. Review of Main Concepts

(a) **Conditional Probability**: $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

(b) **Independence**: Events E and F are independent iff $\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F)$, or equivalently $\mathbb{P}(F) = \mathbb{P}(F|E)$, or equivalently $\mathbb{P}(E) = \mathbb{P}(E|F)$

(c) **Bayes Theorem**: $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$

(d) **Partition:** Nonempty events $E_1, \dots, E_n$ partition the sample space Ω iff

- $E_1, \dots, E_n$ are exhaustive: $E_1 \cup E_2 \cup \dots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$, and
- $E_1, \dots, E_n$ are pairwise mutually exclusive: $\forall i \neq j, E_i \cap E_j = \emptyset$

(e) **Law of Total Probability (LTP)**: Suppose $A_1, \dots, A_n$ partition Ω and let B be any event. Then $\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(\mathsf{B} \mid A_i)\mathbb{P}(A_i)$

(f) **Bayes Theorem with LTP**: Suppose $A_1, \dots, A_n$ partition Ω and let B be any event. Then $\mathbb{P}(A_1|B) = \dfrac{\mathbb{P}(B \mid A_1)\mathbb{P}(A_1)}{\sum_{i=1}^{n} \mathbb{P}(\mathsf{B} \mid A_i)\mathbb{P}(A_i)}$. In particular, $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(\mathsf{B} \mid A)\mathbb{P}(A) + \mathbb{P}(\mathsf{B} \mid A^C)\mathbb{P}(A^C)}$

(g) **Chain Rule**: Suppose $A_1, ..., A_n$ are events. Then,

$$\mathbb{P}(A_1 \cap ... \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3|A_1 \cap A_2)...\mathbb{P}(A_n|A_1 \cap ... \cap A_{n-1})$$

1. Random Grades?

Suppose there are three possible teachers for CSE 312: Martin Tompa, Anna Karlin, and Adam Blank. Suppose the probabilities of getting an A in Martin's class is $\frac{5}{15}$, for Anna's class is $\frac{3}{15}$, and for Adam's class is $\frac{1}{15}$. Suppose you are assigned a grade randomly according to the given probabilities when you take a class from one of these professors, irrespective of your performance. Furthermore, suppose Adam teaches your class with probability $\frac{1}{2}$ and Anna and Martin have an equal chance of teaching if Adam isn't. What is the probability you had Adam, given that you received an A? Compare this to the unconditional probability that you had Adam.

Solution:

Let T, K, B be the events you take 312 from Tompa, Karlin, and Blank, respectively. Let A be the event you get an A.

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A|T)\mathbb{P}(T) + \mathbb{P}(A|K)\mathbb{P}(K) + \mathbb{P}(A|B)\mathbb{P}(B)} = \frac{2}{5+3+2} = \boxed{\frac{1}{5}}$$

2. Game Show

Corrupted by their power, the judges running the popular game show America's Next Top Mathematician have been taking bribes from many of the contestants. During each of two episodes, a given contestant is either allowed to stay on the show or is kicked off. If the contestant has been bribing the judges, she will be allowed to stay with probability 1. If the contestant has not been bribing the judges, she will be allowed to stay with probability 1/3, independent of what happens in earlier episodes. Suppose that 1/4 of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds.

(a) If you pick a random contestant, what is the probability that she is allowed to stay during the first episode?

Solution:

Let S_i be the event that she stayed during the i-th episode. By the Law of Total Probability,

$$\mathbb{P}(S_1) = \frac{1}{4} \cdot 1 + \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{2}$$

(b) If you pick a random contestant, what is the probability that she is allowed to stay during both episodes?

Solution:

By the Law of Total Probability,

$$\mathbb{P}(S_1 \cap S_2) = \frac{1}{4} \cdot 1 \cdot 1 + \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{3}$$

(c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?

Solution:

By the definition of conditional probability and the Law of Total Probability,

$$\mathbb{P}(\overline{S_2} \mid S_1) = \frac{\mathbb{P}(S_1 \cap \overline{S_2})}{\mathbb{P}(S_1)} = \frac{\frac{1}{4} \cdot 1 \cdot 0 + \frac{3}{4} \cdot \frac{1}{3} \cdot \frac{2}{3}}{\frac{1}{2}} = \frac{1/6}{1/2} = \frac{1}{3}$$

(d) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she was bribing the judges?

Solution:

Let B be the event that she bribed the judges. By Bayes' Theorem,

$$\mathbb{P}(B \mid S_1) = \frac{\mathbb{P}(S_1 \mid B)\mathbb{P}(B)}{\mathbb{P}(S_1)} = \frac{1 \cdot \frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

3. Parallel Systems

A parallel system functions whenever at least one of its components works. Consider a parallel system of n components and suppose that each component works with probability p independently.

(a) What is the probability the system is functioning?

Solution:

Let C_i be the event component i is working, and F be the event that the system is functioning. Then, $\mathbb{P}(F) = 1 - \mathbb{P}(F^C) = 1 - \mathbb{P}(\bigcap_{i=1}^n C_i^C) = 1 - \prod_{i=1}^n \mathbb{P}(C_i^C) = 1 - (1-p)^n$.

(b) If the system is functioning, what is the probability that component 1 is working?

Solution:

By Bayes Theorem,

$$\mathbb{P}(C_1|F) = \frac{\mathbb{P}(F|C_1)\mathbb{P}(C_1)}{\mathbb{P}(F)} = \frac{p}{1-(1-p)^n}$$

(c) If the system is functioning and component 2 is working, what is the probability that component 1 is working?

Solution:

$$\mathbb{P}(C_1|C_2, F) = \mathbb{P}(C_1|C_2) = \mathbb{P}(C_1) = p$$

, where the first step is since knowing C_2 and F is just as good as knowing C_2 (since if C_2 happens, F does too), and the second step is by independence.

4. Marbles in Pockets

A girl has 5 blue and 3 white marbles in her left pocket, and 4 blue and 4 white marbles in her right pocket. If she transfers a randomly chosen marble from left pocket to right pocket without looking, and then draws a randomly chosen marble from her right pocket, what is the probability that it is blue?

Solution:

By the Law of Total Probability,

$$\frac{5}{8} \cdot \frac{5}{9} + \frac{3}{8} \cdot \frac{4}{9} = \frac{37}{72}$$

5. Allergy Season

In a certain population, everyone is equally susceptible to colds. The number of colds suffered by each person during each winter season ranges from 0 to 4, with probability 0.2 for each value (see table below). A new cold prevention drug is introduced that, for people for whom the drug is effective, changes the probabilities as shown in the table. Unfortunately, the effects of the drug last only the duration of one winter season, and the drug is only effective in 20% of people, independently.

number of colds	no drug or ineffective	drug effective
0	0.2	0.4
1	0.2	0.3
2	0.2	0.2
3	0.2	0.1
4	0.2	0.0

(a) Sneezy decides to take the drug. Given that he gets 1 cold that winter, what is the probability that the drug is effective for Sneezy?

Solution:

Let E be the event that the drug is effective for Sneezy, and C_i be the event that he gets i colds the first winter. By Bayes' Theorem,

$$\mathbb{P}(E \mid C_1) = \frac{\mathbb{P}(C_1 \mid E)\mathbb{P}(E)}{\mathbb{P}(C_1 \mid E)\mathbb{P}(E) + \mathbb{P}(C_1 \mid \overline{E})\mathbb{P}(\overline{E})} = \frac{0.3 \times 0.2}{0.3 \times 0.2 + 0.2 \times 0.8} = \frac{3}{11}$$

(b) The next year he takes the drug again. Given that he gets 2 colds in this winter, what is the updated probability that the drug is effective for Sneezy?

Solution:

Let the reduced sample space for part (b) be C_1 from part (a). Let D_i be the event that he gets i colds the second winter. By Bayes' Theorem,

$$\mathbb{P}(E \mid D_2) = \frac{\mathbb{P}(D_2 \mid E)\mathbb{P}(E)}{\mathbb{P}(D_2 \mid E)\mathbb{P}(E) + \mathbb{P}(D_2 \mid \overline{E})\mathbb{P}(\overline{E})} = \frac{0.2 \times \frac{3}{11}}{0.2 \times \frac{3}{11} + 0.2 \times \frac{8}{11}} = \frac{3}{11}$$

(c) Why is the answer to (b) the same as the answer to (a)?

Solution:

The probability of two colds whether or not the drug was effective is the same. Hence knowing that Sneezy got two colds does not change the probability of the drug's effectiveness.

6. Infinite Lottery

Suppose we randomly generate a number from the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$. Let A_k be the event we generate the number k, and suppose $\mathbb{P}(A_k) = (\frac{1}{2})^k$. Once we generate a number k, that is the maximum we can win. That is, after generating a value k, we can win any number in $[k] = \{1, ..., k\}$ dollars. Suppose the probability that we win $\$j$ for $j \in [k]$ is "uniform", that is, each has probability $\frac{1}{k}$. Let B be the event we win exactly $\$1$. Given that we win exactly one dollar, what is the probability that the number generated was also 1? You may use the fact that $\sum_{j=1}^{\infty} \frac{1}{j \cdot a^j} = \ln(\frac{a}{a-1})$for $a > 1$.

Solution:

$$\mathbb{P}(A_1|B) = \frac{\mathbb{P}(B|A_1)\mathbb{P}(A_1)}{\sum_{j=1}^{\infty} \mathbb{P}(B|A_j)\mathbb{P}(A_j)} = \frac{\frac{1}{1}\frac{1}{2^1}}{\sum_{j=1}^{\infty} \frac{1}{j}\frac{1}{2^j}} = \frac{1}{2\ln 2} \approx \boxed{0.7213}$$