CSE 312: Foundations of Computing II

Section 2: More Combinatorics, Intro Probability Solutions

0. Review of Main Concept

(a) **Binomial Theorem:** $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

(b) **Principle of Inclusion-Exclusion (PIE):** 2 events: $|A \cup B| = |A| + |B| - |A \cap B|$
3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
In general: +singles - doubles + triples - quads + ...

(c) **Pigeonhole Principle:** If there are n pigeons with k holes and $n > k$, then at least one hole contains at least 2 (or to be precise, $\lceil \frac{n}{k} \rceil$) pigeons.

(d) **Complementary Counting (Complementing):** If asked to find the number of ways to do X, you can: find the total number of ways and then subtract the number of ways to not do X.

(e) **Key Probability Definitions**

  (a) **Sample Space:** The set of all possible outcomes of an experiment, denoted Ω or S
  (b) **Event:** Some subset of the sample space, usually a capital letter such as $E \subseteq \Omega$
  (c) **Union:** The union of two events E and F is denoted $E \cup F$
  (d) **Intersection:** The intersection of two events E and F is denoted $E \cap F$ or EF
  (e) **Mutually Exclusive:** Events E and F are mutually exclusive iff $E \cap F = \emptyset$
  (f) **Complement:** The complement of an event E is denoted E^C or $\overline{E}$ or $\neg E$, and is equal to $\Omega \setminus E$
  (g) **DeMorgan's Laws:** $(E \cup F)^C = E^C \cap F^C$ and $(E \cap F)^C = E^C \cup F^C$
  (h) **Probability of an event E:** denoted $\mathbb{P}(E)$ or $\Pr(E)$ or $P(E)$
  (i) **Partition:** Nonempty events $E_1, \ldots, E_n$ partition the sample space Ω iff
    - $E_1, \ldots, E_n$ are exhaustive: $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$, and
    - $E_1, \ldots, E_n$ are pairwise mutually exclusive: $\forall i \neq j, E_i \cap E_j = \emptyset$
      - Note that for any event A (with $A \neq \emptyset, A \neq \Omega$): A and A^C partition Ω

(f) **Axioms of Probability and their Consequences**

  (a) **Axiom 1: Non-negativity** For any event E, $\mathbb{P}(E) \geq 0$
  (b) **Axiom 2: Normalization** $\mathbb{P}(\Omega) = 1$
  (c) **Axiom 3: Countable Additivity** If E and F are mutually exclusive, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$. Also, if $E_1, E_2, ...$ is a countable sequence of disjoint events, $\mathbb{P}(\bigcup_{k=1}^{\infty} E_i) = \sum_{k=1}^{\infty} \mathbb{P}(E_i)$.
  (d) **Corollary 1: Complementation** $\mathbb{P}(E) + \mathbb{P}(E^C) = 1$
  (e) **Corollary 2: Monotonicity** If $E \subseteq F$, $\mathbb{P}(E) \leq \mathbb{P}(F)$
  (f) **Corollary 2: Inclusion-Exclusion** $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

(g) **Equally Likely Outcomes:** If every outcome in a finite sample space Ω is equally likely, and E is an event, then $\mathbb{P}(E) = \dfrac{|E|}{|\Omega|}$.

  - Make sure to be consistent when counting $|E|$ and $|\Omega|$. Either order matters in both, or order doesn't matter in both.

1. Trick or Treat

Suppose on Halloween, someone is too lazy to keep answering the door, and leaves a jar of exactly N total candies. You count that there are exactly K of them which are kit kats (and the rest are not). The sign says to please take exactly n candies. Each item is equally likely to be drawn. Let X be the number of kit kats we draw (out of n). What is $\Pr(X = k)$, that is, the probability we draw exactly k kit kats?

Solution:

$$\boxed{\Pr(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}}$$

We choose k out of the K kit kats, and $n - k$ out of the $N - K$ other candies. The denominator is the total number of ways to choose n candies out of N total.

2. Staff Photo

Suppose we have 11 chairs (in a row) with 7 TA's, and Professors Karlin, Ruzzo, Rao, and Tompa to be seated. Suppose all seatings are equally likely. What is the probability that every professor has a TA to his/her immediate left and right?

Solution:

Imagine we permute all 7 TA's first – there are $7!$ ways to do this. Then, there are 6 spots between them, in which we choose 4 for the Professors to sit – order matters since each Professor is distinct so we multiply by $4!$. So the total ways is $\binom{6}{4} \cdot 4! \cdot 7!$.

The total number of ways to seat all 11 of us is simply $11!$.

The probability is then $\boxed{\frac{\binom{6}{4} \cdot 4! \cdot 7!}{11!}}$

3. Ingredients

(a) Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent.

Solution:

We use inclusion-exclusion. Let Ω be the set of all anagrams (permutations) of "INGREDIENT", and A_I be the set of all anagrams with two consecutive I's. Define A_E and A_N similarly. $A_I \cup A_E \cup A_N$ clearly are the set of anagrams we don't want. So we use complementing to count the size of $\Omega \setminus (A_I \cup A_E \cup A_N)$. By inclusion exclusion, $|A_I \cup A_E \cup A_N| =$singles-doubles+triples, and by complementing, $|\Omega \setminus (A_I \cup A_E \cup A_N)| = |\Omega| - |A_I \cup A_E \cup A_N|$.

First, $|\Omega| = \frac{10!}{2!2!2!}$ because there are 2 of each of I,E,N's (multinomial coefficient). Clearly, the size of A_I is the same as A_E and A_N. So $|A_I| = \frac{9!}{2!2!}$ because we treat the two adjacent I's as one entity. We also need $|A_I \cap A_E| = \frac{8!}{2!}$ because we treat the two adjacent I's as one entity and the two adjacent E's as one entity (same for all doubles). Finally, $|A_I \cap A_E \cap A_N| = 7!$ since we treat each pair of adjacent I's, E's, and N's as one entity.

Putting this together gives $\boxed{\frac{10!}{2!2!2!} - \left(\binom{3}{1} \cdot \frac{9!}{2!2!} - \binom{3}{2} \cdot \frac{8!}{2!} + \binom{3}{3} \cdot 7!\right)}$

(b) Repeat the question for the letters "AAAAABBB".

Solution:

For the second question, note that no A's and no B's can be adjacent. So let us put the B's down first: $\_B\_B\_B\_$

By the pigeonhole principle, two A's must go in the same slot, but then they would be adjacent, so there are $\boxed{\text{no ways}}$.

4. Fleas on Squares

25 fleas sit on a 5×5 checkerboard, one per square. At the stroke of noon, all jump across an edge (not a corner) of their square to an adjacent square. At least two must end up in the same square. Why?

Solution:

There are two colors on a checkerboard; so 13 are of one color, and 12 are of another.The 13 fleas must jump to the opposite color of which there are only 12 positions, so at least two fleas must land on the same square by the pigeonhole principle.

5. Divide Me

How many numbers in $[360]$ are divisible by:

(a) 4, 6, and 9?

Solution:

This is just the multiples of $\text{lcm}(4, 6, 9) = 36$. There are $\boxed{\frac{360}{36} = 10}$ multiples.

(b) 4, 6, or 9?

Solution:

We must use inclusion-exclusion. $\boxed{\frac{360}{4} + \frac{360}{6} + \frac{360}{9} - \frac{360}{\text{lcm}(4,6)} - \frac{360}{\text{lcm}(4,9)} - \frac{360}{\text{lcm}(6,9)} + \frac{360}{\text{lcm}(4,6,9)}}$

(c) Neither 4, 6, nor 9?

Solution:

This is just the complement of the previous part, so it is 360 minus the answer to (b).

6. Spades and Hearts

Given 3 different spades and 3 different hearts, shuffle them. Compute $\Pr(E)$, where E is the event that the suits of the shuffled cards are in alternating order. What is your sample space?

Solution:

The sample space is all reorderings possible: there are $6!$ such. Now order the spades and hearts independently, so there are $3!^2$ ways to do so. Finally choose whether you want hearts or spades first. So $\Pr(E) = \frac{2 \cdot 3!^2}{6!}$.

For this section, we expect to end here (or before!). The rest of these problems can be done at home for extra practice, or if you finish 1-6 early. Solutions will be posted.

7. Divisibility

Consider the set $T = \{1, 2, ..., 36050\}$, and suppose we choose a subset S of size 3605, each equally likely. What is the probability that there are two (distinct) numbers in S whose difference is divisible by 99?

Solution:

This probability is 1 by the pigeonhole principle. Consider each of the elements of S mod 99 (there are 99 possible remainders $0, ..., 98$). By the pigeonhole principle, since $3605 > 99$, there are at least two with the same remainder. Take those two numbers, and their difference is divisible by 99.

8. Keep Drawing Cards...

How many cards must you draw from a standard 52-card deck (4 suits and 13 cards of each suit) until you are guaranteed to have:

(a) A single pair? (e.g., AA, 99, JJ)

Solution:

The worst that could happen is to draw 13 different cards, but the next is guaranteed to form a pair. So the answer is $\boxed{14}$.

(b) Two (different) pairs? (e.g., AAKK, 9933, 44QQ)

Solution:

The worst that could happen is to draw 13 different cards, but the next is guaranteed to form a pair. But then we could draw the other two of that pair as well to get 16 still without two pairs. So the answer is $\boxed{17}$.

(c) A full house (a triple and a pair)? (e.g., AAAKK, 99922, 555JJ)

Solution:

The worst that could happen is to draw all pairs (26 cards). Then the next is guaranteed to cause a triple. So the answer is $\boxed{27}$.

(d) A straight (5 in a row, with the lowest being A,2,3,4,5 and the highest being 10,J,Q,K,A)?

Solution:

The worst that could happen is to draw all the $A - 4$, $6 - 9$, and $J - K$. After drawing these $11 \cdot 4$ cards, we could still fail to have a straight. Finally, getting a 5 or 10 would give us a straight. So the answer is $\boxed{45}$.

(e) A flush (5 cards of the same suit)? (e.g., 5 hearts, 5 diamonds)

Solution:

The worst that could happen is to draw 4 of each suit (16 cards), and still not have a flush. So the answer is $\boxed{17}$.

(f) A straight flush (a straight but all cards of the same suit)?

Solution:

Same as getting a straight: $\boxed{45}$.

9. Acing the Exams

In a town of 351 students (the number of students, not ones taking CSE 351), every student aces the midterm, final, or both. If 331 of the students ace the midterm and 45 ace the final, how many people who aced the midterm did not ace the final as well?

Solution:

By inclusion-exclusion, the number of people who aced both the midterm and the final is $331 + 45 - 351 = 25$. For one of the 331 students who aced the midterm; either they aced the final or they didn't, so $331 - 25 = 306$ did not ace the final.

10. Congressional Tea Party

Twenty politicians are having a tea party, 6 Democrats and 14 Republicans.

(a) If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans?

Solution:

The sample space is the number of ways to give tea to people, so there are $\binom{20}{10}$ ways. The event is the ways to give tea to only Republicans, of which there are $\binom{14}{10}$ ways. So the probability is $\frac{\binom{14}{10}}{\binom{20}{10}}$.

(b) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 8 Republicans and 2 Democrats?

Solution:

Similarly to the previous part, $\frac{\binom{14}{8}\binom{6}{2}}{\binom{20}{10}}$.