1. Review of Main Concept

(a) **Binomial Theorem:** \( \forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}: (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \)

(b) **Principle of Inclusion-Exclusion (PIE):**

- 2 events: \(|A \cup B| = |A| + |B| - |A \cap B|\)
- 3 events: \(|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|\)

In general: +singles - doubles + triples - quads + . . .

(c) **Pigeonhole Principle:** If there are \( n \) pigeons with \( k \) holes and \( n > k \), then at least one hole contains at least 2 (or to be precise, \( \lceil \frac{n}{k} \rceil \)) pigeons.

(d) **Complementary Counting (Complementing):** If asked to find the number of ways to do \( X \), you can: find the total number of ways and then subtract the number of ways to not do \( X \).

(e) **Key Probability Definitions**

- **Sample Space:** The set of all possible outcomes of an experiment, denoted \( \Omega \) or \( S \)
- **Event:** Some subset of the sample space, usually a capital letter such as \( E \subseteq \Omega \)
- **Union:** The union of two events \( E \) and \( F \) is denoted \( E \cup F \)
- **Intersection:** The intersection of two events \( E \) and \( F \) is denoted \( E \cap F \) or \( EF \)
- **Mutually Exclusive:** Events \( E \) and \( F \) are mutually exclusive iff \( E \cap F = \emptyset \)
- **Complement:** The complement of an event \( E \) is denoted \( E^C \) or \( \overline{E} \) or \( \neg E \), and is equal to \( \Omega \setminus E \)
- **DeMorgan’s Laws:** \((E \cup F)^C = E^C \cap F^C \) and \((E \cap F)^C = E^C \cup F^C \)
- **Probability of an event:** \( E \): denoted \( \mathbb{P}(E) \) or \( \text{Pr}(E) \) or \( P(E) \)
- **Partition:** Nonempty events \( E_1, \ldots, E_n \) partition the sample space \( \Omega \) iff
  - \( E_1, \ldots, E_n \) are exhaustive: \( E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega \), and
  - \( E_1, \ldots, E_n \) are pairwise mutually exclusive: \( \forall i \neq j, E_i \cap E_j = \emptyset \)
    - Note that for any event \( A \) (with \( A \neq \emptyset, A \neq \Omega \)): \( A \) and \( A^C \) partition \( \Omega \)
- **Axioms of Probability and their Consequences**
  - **Axiom 1:** **Non-negativity** For any event \( E \), \( \mathbb{P}(E) \geq 0 \)
  - **Axiom 2:** **Normalization** \( \mathbb{P}(\Omega) = 1 \)
  - **Axiom 3:** **Countable Additivity** If \( E \) and \( F \) are mutually exclusive, then \( \mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) \).
    Also, if \( E_1, E_2, \ldots \) is a countable sequence of disjoint events, \( \mathbb{P}(\bigcup_{k=1}^{\infty} E_i) = \sum_{k=1}^{\infty} \mathbb{P}(E_i) \).
  - **Corollary 1:** **Complementation** \( \mathbb{P}(E) + \mathbb{P}(E^C) = 1 \)
  - **Corollary 2:** **Monotonicity** If \( E \subseteq F \), \( \mathbb{P}(E) \leq \mathbb{P}(F) \)
  - **Corollary 2:** **Inclusion-Exclusion** \( \mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F) \)
- **Equally Likely Outcomes:** If every outcome in a finite sample space \( \Omega \) is equally likely, and \( E \) is an event, then \( \mathbb{P}(E) = \frac{|E|}{|\Omega|} \).
  - Make sure to be consistent when counting \(|E|\) and \(|\Omega|\). Either order matters in both, or order doesn’t matter in both.
2. Trick or Treat
Suppose on Halloween, someone is too lazy to keep answering the door, and leaves a jar of exactly \( N \) total candies. You count that there are exactly \( K \) of them which are kit kats (and the rest are not). The sign says to please take exactly \( n \) candies. Each item is equally likely to be drawn. Let \( X \) be the number of kit kats we draw (out of \( n \)). What is \( \Pr(X = k) \), that is, the probability we draw exactly \( k \) kit kats?

3. Staff Photo
Suppose we have 11 chairs (in a row) with 7 TA’s, and Professors Karlin, Ruzzo, Rao, and Tompa to be seated. Suppose all seatings are equally likely. What is the probability that every professor has a TA to his/her immediate left and right?

4. Ingredients
(a) Find the number of ways to rearrange the word “INGREDIENT”, such that no two identical letters are adjacent to each other. For example, “INGREEDINT” is invalid because the two E’s are adjacent.

(b) Repeat the question for the letters “AAAAABBB”.

5. Fleas on Squares
25 fleas sit on a \( 5 \times 5 \) checkerboard, one per square. At the stroke of noon, all jump across an edge (not a corner) of their square to an adjacent square. At least two must end up in the same square. Why?

6. Divide Me
How many numbers in \([360]\) are divisible by:
   (a) 4, 6, and 9?
   (b) 4, 6, or 9?
   (c) Neither 4, 6, nor 9?

7. Spades and Hearts
Given 3 different spades and 3 different hearts, shuffle them. Compute \( \Pr(E) \), where \( E \) is the event that the suits of the shuffled cards are in alternating order. What is your sample space?

For this section, we expect to end here (or before!). The rest of these problems can be done at home for extra practice, or if you finish 1-6 early. Solutions will be posted.

8. Divisibility
Consider the set \( T = \{1, 2, \ldots, 36050\} \), and suppose we choose a subset \( S \) of size 3605, each equally likely. What is the probability that there are two (distinct) numbers in \( S \) whose difference is divisible by 99?

9. Keep Drawing Cards...
How many cards must you draw from a standard 52-card deck (4 suits and 13 cards of each suit) until you are guaranteed to have:
   (a) A single pair? (e.g., AA, 99, JJ)
   (b) Two (different) pairs? (e.g., AAKK, 9933, 44QQ)
(c) A full house (a triple and a pair)? (e.g., AAAKK, 99922, 555JJ)

(d) A straight (5 in a row, with the lowest being A,2,3,4,5 and the highest being 10,J,Q,K,A)?

(e) A flush (5 cards of the same suit)? (e.g., 5 hearts, 5 diamonds)

(f) A straight flush (a straight but all cards of the same suit)?

10. Acing the Exams
In a town of 351 students (the number of students, not ones taking CSE 351), every student aces the midterm, final, or both. If 331 of the students ace the midterm and 45 ace the final, how many people who aced the midterm did not ace the final as well?

11. Congressional Tea Party
Twenty politicians are having a tea party, 6 Democrats and 14 Republicans.  
(a) If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans?

(b) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 8 Republicans and 2 Democrats?