CSE 312: Foundations of Computing II

Combinatorics 1 Solutions

Review of Main Concepts

(a) **Product Rule:** Suppose events $A_1, ..., A_n$ each have $m_1, ...m_n$ possible outcomes, respectively. Then there are $m_1 \cdot m_2 \cdot m_3 \cdots m_n = \prod_{i=1}^{n} m_i$ possible outcomes overall.

(b) **Number of ways to order n distinct objects:** $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$

(c) **Number of ways to select from n distinct objects:**

  (a) **Permutations** (number of ways to linearly arrange k objects out of n distinct objects, when the order of the k objects matters):

$$P(n, k) = \frac{n!}{(n-k)!}$$

  (b) **Combinations** (number of ways to choose k objects out of n distinct objects, when the order of the k objects does not matter):

$$\frac{n!}{k!(n-k)!} = \binom{n}{k} = C(n, k)$$

(d) **Multinomial coefficients:** Suppose there are n objects, but only k are distinct, with $k \leq n$. (For example, "godoggy" has $n = 7$ objects (characters) but only $k = 4$ are distinct: (g, o, d, y)). Let n_i be the number of times object i appears, for $i \in \{1, 2, \ldots, k\}$. (For example, $(3, 2, 1, 1)$, continuing the "godoggy" example.) The number of distinct ways to arrange the n objects is:

$$\frac{n!}{n_1! n_2! \cdots n_k!} = \binom{n}{n_1, n_2, \ldots, n_k}$$

(e) **Pigeonhole Principle** Suppose there are $n - 1$ pigeon holes and n pigeons, and each pigeon goes into a hole. Then, there must be some hole that has two pigeons in it. This simple observation is surprisingly useful in computer science.

We can put this more generally as: if there are n pigeons and k holes, and $n > k$, some hole has at least $\lceil \frac{n}{k} \rceil$ pigeons.

For the pigeon haters out there, we can also express this as "we have n holes and $n - 1$ pigeons...". Pick your favorite.

Birthday Cake

A chef is preparing desserts for the week, starting on a Sunday. On each day, only one of five desserts (apple pie, cherry pie, strawberry pie, pineapple pie, and cake) may be served. On Thursday there is a birthday, so cake must be served that day. On no two consecutive days can the chef serve the same dessert. How many dessert menus are there for the week?

Solution:

Start from Thursday and work forward and backward in the week: $4 \cdot 4 \cdot 4 \cdot 4 \cdot 1 \cdot 4 \cdot 4 = \boxed{4^6}$

HBCDEFGA

How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

Solution:

First, we assign A not to the beginning. There are two cases:

Case 1: A is at the end

In this case, the remaining 7 letters can go anywhere, so there are $7!$ arrangements.

Case 2: A is neither at the end nor the beginning

There are 6 choices for A if it is not at either end. Then, there are 6 remaining valid spots for H (cannot be at the end). Finally the other 6 letters can be arranged in any way. So we have $6^2 \cdot 6!$ ways.

Hence the total number of ways is $\boxed{7! + 6^2 \cdot 6!}$.

We can approach this another way, using the Principle of Inclusion and Exclusion. There are $8!$ ways to order them. Subtract $7!$ for those with A at the beginning and $7!$ for those with H at the end. Then add back $6!$ for those with A at the beginning and H at the end, leaving us with $\boxed{8! - 2 \cdot 7! + 6!}$.

Full Class

There are 40 seats and 40 students in a classroom. Suppose that the front row contains 10 seats, and there are 5 students who must sit in the front row in order to see the board clearly. How many seating arrangements are possible with this restriction?

Solution:

$\boxed{\binom{10}{5} \cdot 5! \cdot 35!}$ Seat the students who must sit in the front row first. There are $\binom{10}{5} \cdot 5!$ ways to assign seats to those students, since we choose any 5 of the 10 seats, and then assign them. Then there are 35 students and 35 seats left, so there are $35!$ ways to assign seats to the other students

Friendly Proofs

First, assume that if Alice is friends with Bob, Bob must also be friends with Alice. In other words, there are no unrequited friendships.

Given this, prove that in any group of people (of size at least 2), there are two people with the same number of friends in the group.

Solution:

Say there are k people with 0 friends in the group. If $k \geq 2$, we are done. Otherwise, the remaining $n - k$ people can have between 1 and $n - k - 1$ friends. The statement follows by the Pigeonhole Principle.

Paired Finals

Suppose you are to take the CSE 312 final in pairs. There are 100 students in the class and 8 TAs, so 8 lucky students will get to pair up with a TA. Each TA must take the exam with some student, but two TAs cannot take the exam together. How many ways can they pair up?

Solution:

First we choose the 8 lucky students, and then pair them with a TA. There are $\binom{100}{8}$ ways to choose the students, and $8!$ ways to pair them with the TAs. There are 92 students left. The first one has 91 choices. Then there are 90 students left. The next one has 89 choices. And so on. So the total number of ways is $\boxed{\binom{100}{8} \cdot 8! \cdot 91 \cdot 89 \cdot \ldots \cdot 3 \cdot 1}$.

Escape the Professor

There are 6 security professors and 7 theory professors taking part in an escape room. If 4 security professors and 4 theory professors are chosen and paired off, how many pairings are possible?

Solution:

$\boxed{\binom{6}{4}\binom{7}{4}4!}$. First choose 4 of the security professors, then 4 of the theory professors. Then assign each theory professor to a security professor.

Photographs

Suppose that 8 people, including you and a friend, line up for a picture. In how many ways can the photographer organize the line if she wants to have fewer than 2 people between you and your friend?

Solution:

There are two cases; so, we count using the rule of sum:

Case 1: You are next to your friend. Then there are there are 7 different slots. Then, there are 7 sets of positions you and your friend can occupy (positions 1/2, 2/3, ..., 7/8), and for each set of positions, there are 2 ways to arrange you and your friend, So there are $7 \cdot 2$ ways to pick positions for you and your friend.

Case 2: There is exactly 1 person between you and your friend. Then, there are 6 sets of positions you and your friend can occupy (positions 1/3, 2/4, ... , 6/8), and for each set of positions, there are 2 ways to arrange you and your friend. So there are $6 \cdot 2$ ways to pick positions for you and your friend.

Note that in both cases, there are then $6!$ ways to arrange the remaining people, so we multiply both cases by $6!$.

Therefore, the answer is $\boxed{(2 \cdot 7 + 2 \cdot 6) \cdot 6!}$

Rabbits!

Rabbits Peter and Pauline have three offspring: Flopsie, Mopsie, and Cotton-tail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

Solution:

If Peter and Pauline go to the same store, there are 4 stores it could be. For each such choice, there are 3 choices of store for each of the 3 offspring, so 3^3 choices for all the offspring. If Peter and Pauline go to different stores, there are $4 \cdot 3 = 12$ pairs of stores they could go to. For each such choice, there are 2 choices of store for each of the 3 offspring, so 2^3 choices for all the offspring. Therefore the answer is $\boxed{4 \cdot 3^3 + 12 \cdot 2^3}$.

Seating

How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if . . .

(a) . . . all couples are to get adjacent seats?

Solution:

$\boxed{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2 = 2^5 \cdot 5!}$: there are $5!$ permutations of the 5 couples, and then 2 permutations within each of the 5 couples.

(b) . . . the seats are assigned arbitrarily, except that one couple insists on not sitting in adjacent seats?

Solution:

There are $9! \cdot 2$ arrangements in which this couple does sit in adjacent seats, since you can treat the couple as a ninth unit added to the other 8 individuals, and then there are 2 permutations of that couple's seats. That means the answer to the question is $\boxed{10! - 9! \cdot 2 = 8 \cdot 9!}$.

Alternatively, we can do casework. Name the two people in the couple A and B. There are two cases: A can sit on one of the ends, or not. If A sits on an end seat, A has 2 choices and B has 8 possible seats. If A doesn't sit on the end, A has 8 choices and B only has 7. So there are a total of $2 \cdot 8 + 8 \cdot 7$ ways A

and B can sit. Once they do, the other 8 people can sit in $8!$ ways since there are no other restrictions. Hence the total number of ways is $\boxed{(2 \cdot 8 + 8 \cdot 7)8! = 9 \cdot 8 \cdot 8! = 8 \cdot 9!}$.

Weird Card Game

In how many ways can a pack of fifty-two cards be dealt to thirteen players, four to each, so that every player has one card of each suit?

Solution:

Deal one suit at a time. For each suit, there are $13!$ ways to distribute one card to each person. So the answer is $\boxed{13!^4}$.

Extended Family Portrait

A group of n families, each with m members, are to be lined up for a photograph. In how many ways can the nm people be arranged if members of a family must stay together?

Solution:

First order the families, of which there are $n!$ ways. Within each family, there are $m!$ ways to order their members. So there are a total of $\boxed{n!(m!)^n}$ ways.