Directions:
When asked for a short answer (such as a single number), also show and explain your work briefly and clearly. Unless you are asked to, you should leave your answer in terms of factorials, combinations, etc., for instance $26^7$ or $26!/7!$ or $26 \cdot \binom{26}{7}$.

Your solutions need to be concise and clear. We will take off points for lack of clarity or for excess verbosity. Please see section worksheet solutions (posted on the course website) to gauge the level of detail we are expecting.

Remember to assign your PDF pages to questions when submitting your solutions on Gradescope. Note that one can assign multiple pages to a single question and a single page to multiple questions (if one solves 2 questions on the same page, which we would strongly prefer that you do not do) We will take off points if you do not properly assign PDF pages to questions.

Please put your final answers to questions in boxes. (This is easy if you use the latex template, which is highly recommended).

1. Sample Spaces and Probabilities (24 points)
For each of the following scenarios first answer the following two questions and then answer the question stated.
(i) What is the sample space and how big is it? (ii) What is the probability of each outcome in the sample space?
(a) [4 Points] You flip a fair coin 50 times. What is the probability of exactly 20 heads?
(b) [4 Points] You roll 2 fair dice. What is the probability that the sum of the two values showing is 4?
(c) [4 Points] You are given a random 5 card poker hand (selected from a single deck). What is the probability you have a full-house (3 cards of one rank and 2 cards of another rank)?
(d) [4 Points] 20 labeled balls are placed into 10 labeled bins (with each placement equally likely). What is the probability that bin 1 contains 3 balls?
(e) [4 Points] There are 30 psychiatrists and 24 psychologists attending a certain conference. Three of these 54 people are randomly chosen to take part in a panel discussion. What is the probability that at least one psychologist is chosen? What is the probability that exactly three psychologists are chosen?
(f) [4 Points] You buy ten cupcakes choosing from 3 different types (chocolate, vanilla and caramel). Cupcakes of the same type are indistinguishable. What is the probability that you have at least one of each type?

2. Miscounting (10 points)
Consider the question: what is the probability of getting a 6-card poker hand that contains a card from all four suits (order doesn’t matter):
Each of the $\binom{52}{6}$ hands is equally likely. Let $E$ be the event that the hand selected contains a card from all four suits. Then
$$\Pr(E) = \frac{|E|}{\binom{52}{6}}$$
For $|E|$, first pick one heart, then one spade, then one diamond, then one clubs, and, finally, two additional cards. Therefore
$$|E| = 13^4 \cdot \binom{48}{2}$$
and hence
$$\Pr(E) = \frac{13^4 \cdot \binom{48}{2}}{\binom{52}{6}}.$$
Explain what is wrong with this solution. If there is over-counting in $|E|$, characterize all hands that are counted more than once, and how many times each such hand is counted. Also, give the correct answer for $\Pr(E)$.

3. Strange Probabilities (10 points)
Probability can be strange. Suppose you have 3 dice: Die A has 3 sides with the numbers 2, 4 and 9 on it. Die B has 3 sides with the numbers 1, 6 and 8. Die C has 3 sides with the numbers 3, 5, and 7 on it. (Let’s not worry about any physical impossibilities here.) You toss all 3 dice and each outcome is equally likely. What is the probability that die A shows a larger number than die B? What is the probability that die B shows a larger number than die C? What is the probability that die C shows a larger number than die A?

4. Random Questions (15 points)
(a) [5 Points] What is the probability that the digit 1 doesn’t appear among $n$ digits where each digit is one of (0-9) and all sequences are equally likely?
(b) [5 Points] Suppose you randomly permute the numbers 1, 2, . . . , $n$, that is, you select a permutation uniformly at random. What is the probability that the number $k$ ends up in the $i$-th position in the resulting permutation?
(c) [5 Points] A fair coin is flipped $n$ times (each outcome in $\{H, T\}^n$ is equally likely). What is the probability that all heads occur at the end of the sequence? (The case that there are no heads is a special case of having all heads at the end of the sequence, i.e. 0 heads.)

5. Hat Shooting (12 points)
Three people are brought into a room. A hat is placed on each person’s head. The hat is equally likely to be Red or Blue. (So each of the 8 possibilities is equally likely.) Each person sees the colors of the other people’s hats, but not their own. Each person, without communication, writes down one of the following: "My hat is red", "My hat is blue" or "Pass". All three people will be put in jail unless (a) at least one of them doesn’t pass, and (b) everyone who doesn’t pass is right about his/her own hat color. Importantly, they can agree ahead of time on a strategy, with the hopes of not going to prison.
(a) [4 Points] What is the probability that they are not sent to prison if each person guesses the color of his/her own hat?
(b) [4 Points] What is the probability that they are not sent to prison if two of them pass and 1 of them guesses?
(c) [4 Points] What is the probability that they are not sent to prison if they use the following strategy: Each person looks at the other two hats. If they are both blue, then the person guesses red. If they are both red, then the person guesses blue. If they are different, the person passes.

6. Weighted Die (9 points)
Consider a weighted die such that
- $\Pr(1) = \Pr(2)$,
- $\Pr(3) = \Pr(4) = \Pr(5) = \Pr(6)$, and
- $\Pr(1) = 3 \Pr(3)$.
What is the probability that the outcome is 3 or 4?
7. Switching envelopes (22 points)
You are shown two envelopes and told the following facts:

- Each envelope has some number of dollars in it, but you don’t know how many.
- The amount in the first envelope is different from the amount in the second.
- Although you don’t know exactly how much money is in each envelope, you are told that it is an integer number of dollars that is at least 1 and at most 100.
- You are told that you can pick an envelope, look inside, and then you will be given a one-time option to switch envelopes (without looking inside the new envelope). You will then be allowed to keep the money in envelope you end up with.

Your strategy is the following:

(a) You pick an envelope uniformly at random.
(b) You open it and count the amount of money inside. Say the result is \(x\).
(c) You then, independently, select an integer \(y\) between 1 and 100 uniformly at random.
(d) If \(y > x\), you switch envelopes, otherwise you stay with the envelope you picked in step (a)

Show that you have a better than 50-50 chance of taking home the envelope with the larger amount of money in it. More specifically, suppose the two envelopes have \(i\) and \(j\) dollars in them respectively, where \(i < j\). Calculate the probability that you take home the envelope with the larger amount of money.

8. Which bags? (8 points)
The 8 points are for parts (a) and (b). Extra credit is considered separately.
The names of 100 people are placed into 100 closed bags, one name per bag, and the bags are lined up on a table in a room. One by one the people are led into the room; each may look in at most 50 bags, but must leave the room exactly as they found it. From the moment a person is led into the room, that person is not allowed any further communication with the others.
The people have a chance to plot their strategy in advance (and come up with a coordinated strategy if they so desire) and they are going to need it, because unless every single person finds their own name among the 50 bags they look in, all of them will be put in jail!

(a) What is the probability they are not put in jail if they all look in the same 50 boxes?

(b) What is the probability they are not put in jail if each person, independently, looks in a random set of 50 boxes?

(c) Extra Credit: (hard without a hint) Come up with a strategy for the people that guarantees that they won’t be be put in jail with probability at least 0.3. (In the next homework, I will walk you through the development of the answer to this question, but I thought I’d first give you a chance to ponder it.)