NAME: .......................................................... Solutions

Student ID number: ..........................................................

Section (Number or TA Name or Time): ..............................

DIRECTIONS:

• Fill out the information above.
• Closed book, closed notes except for one 8.5×11 sheet.
• Time limit 50 minutes.
• Calculators not allowed (or needed).
• Feel free to scribble on the back sides of the sheets.
• Unless you are specifically asked to, you do not need to justify your answers.
• If you see a problem you are not sure how to solve, I recommend going on to the next problem and coming back to it later.
• Do not turn the page until I tell you to.
• Good luck!
There are three coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. When one of the three coins is selected uniformly at random and flipped, it shows heads. What is the probability that it was the two-headed coin? In other words, what is

\[ P_r(\text{two-headed selected} | \text{flip shows heads}) \]?

\[
\Pr(\text{2-headed} | \text{H}) = \frac{\Pr(\text{H} | \text{2-headed}) \Pr(\text{2-headed})}{\Pr(\text{H})}
\]

\[
\Pr(\text{H}) = \Pr(\text{H} | \text{2-headed}) \frac{1}{3} + \Pr(\text{H} | \text{fair}) \frac{1}{2} + \Pr(\text{H} | \text{biased}) \frac{3}{4}
\]

\[
= \frac{1}{3} \left[ 1 + \frac{1}{2} + \frac{3}{4} \right] = \frac{1}{3} \cdot \frac{9}{4}
\]

\[
\Pr(\text{2-headed} | \text{H}) = \frac{1 \cdot \frac{1}{3} \cdot \frac{9}{4}}{\frac{1}{3} \cdot \frac{9}{4}} = \frac{4}{9}
\]
2. (30 points) Circle True (T) or False (F). (No justification required.)

Keep in mind that True means **always** true.

False means that there are situations in which the statement is false.

**3 points each, -1 for each incorrect answer.**

(a) If you toss a coin independently 100 times, with probability 3/4 of coming up heads each time, the probability that you see exactly 10 heads is $(3/4)^{10}(1/4)^{90}$................................................. T F

(b) Suppose that $k$ is a positive integer. If $X$ is a geometric random variable with parameter $p$, then $Pr(X > k) = (1 - p)^k$.............................................................. T F

(c) For any random variables $X$ and $Y$, if $a, b, c$ are all constants, then $Var(aX + bY + c) = a^2Var(X) + b^2Var(Y)$.............................................................. T F

(d) If $X$ is a Poisson random variable with parameter $\lambda$, then $2X = X + X$ is a Poisson random variable with parameter $2\lambda$.............................................................. T F

(e) Suppose that $Z = XY$ where $X$ and $Y$ are independent Bernoulli random variables with parameter $p$. Then $Z$ is Bernoulli with parameter $p^2$.............................................................. T F

(f) The number of ways to choose 8 cards from a 52 card deck that constitute 4 distinct pairs (e.g. two 3’s, two 6’s, two Jacks and two Kings) is $\binom{13}{2}\binom{4}{2}$.............................................................. T F

(g) The number of paths from (0,0) to (100,100) that do not go through (75, 50) is $\binom{200}{100} - \binom{125}{50} \cdot \binom{75}{50}$.(Every step increments one coordinate and the other remains unchanged.) ................................................. T F

(h) Suppose that there are 15 distinct toppings I can put on my pizza and I can choose any subset of these toppings (including none). The number of ways I can choose 3 different pizzas, one for each of my three children (Alex, Bob and Carol) is $2^{15} \cdot (2^{15} - 1) \cdot (2^{15} - 2)$.............................................................. T F

(i) The number of ways to arrange 20 distinct charms (e.g. jewels of different colors) on a bracelet is 20! (The arrangement is not changed by rotating the bracelet.).............................................................. T F

(j) A team of 10 players is selected at random out of 18 possible people, where 15 are women and 3 are men. Then the probability that the team selected has at least one man is $1 - \frac{\binom{15}{10}\binom{18}{10}}{\binom{18}{10}}$.............................................................. T F
3. (20 points)

(a) Suppose we have a bag with 100 balls in it. Of these, 60 balls are white and the rest are black. Suppose that we draw 30 balls from the bag without replacement uniformly at random. What is the probability that there are 20 white balls among the 30 balls drawn?

\[ P(\text{exactly 20}) = \frac{\binom{60}{20} \cdot \binom{40}{10}}{\binom{100}{30}} \]

or

\[ \text{Prob at least 20} = \sum_{k=20}^{30} \frac{\binom{60}{k} \cdot \binom{40}{30-k}}{\binom{100}{30}} \]

We accepted both answers due to possible ambiguity.

(b) What is the probability that it takes 50 tosses to see 5 heads in independent coin tosses with probability 0.1 of getting a heads?

\[ \text{Probability that 50th toss is the fifth heads} = \binom{49}{4} (0.1)^5 (0.9)^{45} \]

(c) For the midterm, each of the 130 students in the class is independently told to take the midterm in room A with probability 2/3 or room B otherwise. Out of the \(\binom{130}{2}\) pairs of students, exactly \(K\) pairs of students in the class are friends with each other. Use linearity of expectation to determine the expected number of pairs of friends that take the midterm in the same room.

\[ X_i = \begin{cases} 1 & \text{if ith pair of friends take midterm in same room} \\ 0 & \text{otherwise} \end{cases} \]

\[ \text{Exp \# pairs among K that take midterm in same room} = \mathbb{E}(\sum_{i=1}^{K} X_i) = \sum_{i=1}^{K} \mathbb{E}(X_i) \]

\[ \mathbb{E}(X_i) = \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{5}{9} \]

Answer: \(K \cdot \frac{5}{9}\)
On a typical day in CSE 312, the number of students that show up is well approximated by a Poisson random variable with parameter 60. Each student that shows up independently decides to actually pay attention in class with probability 0.6. Use the law of total probability to determine the probability that on a typical day, exactly 50 students are paying attention (and the rest aren't). Leave your answer in the form of a sum.

\[
\begin{align*}
X & \quad \# \text{ of students that show up} \quad X \sim \text{Poi}(60) \\
Y & \quad \# \text{ of students that pay attention} \\
\Pr(Y = k \mid X = n) & = \binom{n}{k} 0.6^k 0.4^{n-k} \\
\Pr(Y = 50) & = \sum_{n=50}^{\infty} \Pr(Y = 50 \mid X = n) \Pr(X = n) \\
& = \sum_{n=50}^{\infty} \binom{n}{50} 0.6^{50} 0.4^{n-50} e^{-60} \frac{60^n}{n!}
\end{align*}
\]
5. (16 points)

(a) What is $E \left( \frac{X^2}{5} \right)$ if $X$ is Poisson with parameter 5?

Your answer should be a single number.

$$\text{Var}(X) = E(X^2) - (E(X))^2 \implies E(X^2) = \text{Var}(X) + (E(X))^2$$

$$= 5 + 5^2$$

$$= 30$$

for Poisson(5)

(b) Suppose I repeatedly and independently draw a card from a standard 52 card deck. (The draws are with replacement.) What is the expected number of draws until I see all 4 suits? Your answer should be a single number.

$T_1$  # draws to see any suit  (=1)

$T_2$  # draws after 1st draw to see different suit

$\sim Geo \left( \frac{3}{4} \right)$

$T_3$  # draws after first seeing 2nd suit to see third suit

$\sim Geo \left( \frac{1}{2} \right)$

$T_4$  # draws after first seeing 3rd suit to see 4th

$\sim Geo \left( \frac{1}{4} \right)$

$E(\text{time to see all 4 suits})$

$= E(T_1) + E(T_2) + E(T_3) + E(T_4)$

$= 1 + \frac{4}{3} + 2 + 4$

$= 8 \frac{1}{3}$